Final Exam General Relativity

February 4, 2011

- ► Make every exercise on a separate sheet of paper.
- ► Write your name and student number on every sheet.
- ► Write clearly!
- ► You are allowed to use the lecture notes only.
- ► Divide your time wisely over the exercises.

1 Schwarzshild metric

Consider the Schwarzshild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Let us assume an astronaut passes the event horizon of a black hole at a certain (proper) time. He will fall into the singularity inevitably, but how long can he survive?

(1) Show that, once he has passed the event horizon his radial speed obeys the following inequality:

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

(2) Show that the inequality becomes an equality if he does not have any angular motion and if $E \to 0$, where E is defined as the quantity arising in the geodesic equation for t, which reads $dE/d\tau = 0$.

(3) Calculate the maximal proper time you can spend inside the event horizon of a black hole, before you hit the singularity. Restore units of c in the end. (Dimension of G is $[G] = m^3kg^{-1}s^{-2}$). It may surprise you that for supermassive black holes it will take you a few hours to hit the singularity. (Hint: try a variable substitution $r \to \alpha \sin^2\theta$)

(4) Draw the Penrose diagram corresponding to the Scwarzshild metric (indicate where horizons and singularities are present) and draw two trajectories:
1) The trajectory we just calculated, i.e. the trajectory of an observer falling into the black hole with $E = 0$.
2) A trajectory of an observer falling into the black hole with a nonzero energy.
Conclude that it is physically impossible to fall into the black hole (pass the event horizon) with zero energy. So you will always hit the horizon within the time calculated in part 3.
IMPORTANT: Explain your drawings and give arguments why your trajectories look the way they do. For example: What characterizes the $E = 0$ trajectory?

2 FRW universe

In this problem we will study the flat FRW universe, whose metric is given by

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \equiv -dt^2 + a^2(t)\delta_{jk}dx^j dx^k.$$

Recall that

$$\begin{aligned} R_{tt} &= -3\frac{\ddot{a}}{a}, \\ R_{ij} &= \delta_{ij}(a\ddot{a} + 2\dot{a}^2), \\ R &= 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \end{aligned}$$

We consider a single-component perfect fluid with equation of state $p = w\rho$ and stress energy tensor given by

$$T^{\mu}{}_{\nu} = \rho\,\mathrm{diag}\left(-1, w, w, w\right).$$

(1) Derive the Christoffel symbols and the Einstein equations (for a nonzero cosmological constant, $G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$).

(2) Show that the vanishing divergence of $T^{\mu\nu}$, ($D_\mu T^{\mu\nu} = 0$), implies

$$\frac{d}{da}(\rho a^3) = -3pa^2, \tag{1}$$

where $p = w\rho$ is the pressure of a perfect fluid.

(3) Use eq. (1) to show that for a relativistic fluid, for which $w = 1/3$, the density is

$$\rho = \frac{3\beta^2}{8\pi G} a(t)^{-4},$$

with β an integration constant.

(4) Show that, given the result of part (3), the Einstein equations are solved by the solution

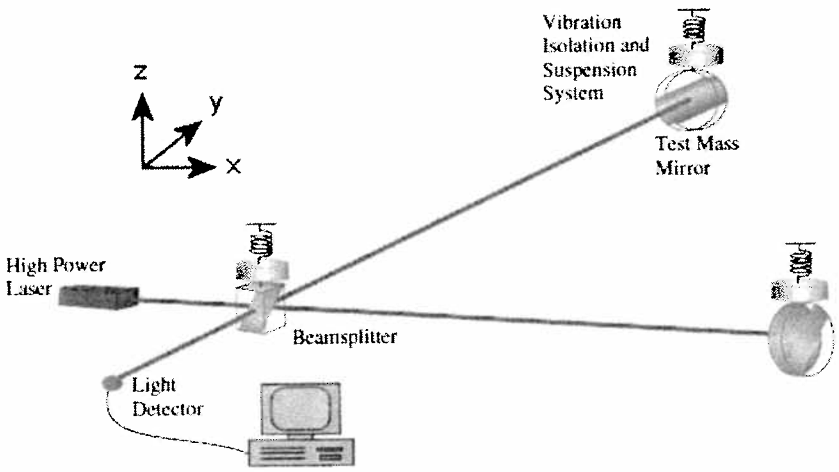
$$a(t)^2 = \beta\sqrt{\frac{3}{\Lambda}} \sinh\left(2t\sqrt{\frac{\Lambda}{3}}\right).$$

(5) Show that $a(t) \propto t^\gamma$ for very early times ($t \to 0$) and that the prefactor does not depend on Λ. Hence, for early times, Λ has no effect on the time dependence of the scale factor $a(t)$. Determine $a(t)$ in this limit and show that, for these early times, the result is the same as we would have got by putting $\Lambda = 0$ from the start.

(6) Consider late times, $t_l \gg 1/\sqrt{\Lambda}$, such that $a(t)^2 \approx \beta\sqrt{\frac{3}{4\Lambda}} e^{2t\sqrt{\Lambda/3}}$. In this regime the cosmological constant dominates the time dependence of the scale factor, $\Lambda \gg \rho$. Show that this result is indeed obtained by putting $\rho = 0$ in the Einstein equations (you can no longer fulfill the boundary condition $a(t) \to 0$ for a finite t in this limit).

(7) Show that a photon traveling along the x-axis, will travel only a finite distance in this universe, i.e. $x(\infty) - x(t_i) =$ finite, for every $t_i \geq 0$.

3 Gravitational waves



Schematic diagram of a Michelson interferometer for use as a gravitational wave detector

(1) Explain how the device in the figure can be used to detect gravitational waves.

(2) Describe the metric tensor $g_{\mu\nu}$ for a polarized gravitational wave coming in from the direction $-x$, $-y$, or $-z$. Which of these waves will the detector observe and which not? Will the detector tell them apart?