Retake Exam General Relativity

March 18, 2011

- ▶ Make every exercise on a separate sheet of paper.
- ▶ Write your name and student number on every sheet.
- ▶ Write clearly!
- ▶ You are allowed to use lecture notes only.
- ▶ Divide your time wisely over the exercises.
- ▶ $c = 1$

Problem 1: Rotating black holes (35 points)

The Kerr-Newman black hole is an exact solution of the Einstein field equations possessing mass, angular momentum, and (in principle but not in astrophysical cases) charge. The metric in Boyer-Lindquist coordinates is:

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2GMr - Q^2}{\rho^2}\right)\mathrm{d}t^2 - \frac{(2GMr - Q^2)2a\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi \\ &+ \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{(2GMr - Q^2)2a\sin^2\theta}{\rho^2}\right)\sin^2\theta\mathrm{d}\phi^2 \ , \end{aligned} \tag{1}$$

where

$$a^2 + Q^2 \leq G^2M^2 \ , \tag{2}$$

$$M \equiv \text{mass}, \quad Q \equiv \text{charge}, \quad a = \frac{J}{GM} \equiv \text{angular momentum per unit mass},$$

$$\Delta \equiv r^2 - 2GMr + a^2 + Q^2 \ , \qquad \rho^2 \equiv r^2 + a^2\cos^2\theta \ .$$

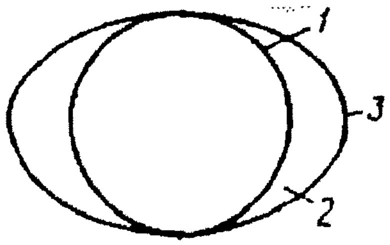

Figure 1: Kerr spacetime

1. The above metric can describe all the asymptotically flat black hole spacetimes studied in the lectures. Can you make a list of the different possible black holes depending on the values of the asymptotic quantities M, Q, J? What is the difference in each case between inequality (2) being saturated or not? Why do we require that this inequality be satisfied?

2. Do you expect the Kerr-Newman metric to describe a black hole when we set $M = Q = J = 0$, but we keep a to be nonzero? How does the metric look like in this case and what is the curvature (Riemann) tensor? What is your conclusion for the resulting spacetime?

3. Let us consider the chargeless case of metric (1). Can you name the region 2 and the boundaries 1 and 3 on Fig. 1? What is the region enclosed within surface 1? Explain the physical properties differentiating region 2 from the rest of space. Which equations describe surfaces 1 and 3 and what do they physically mean?

Problem 2: Black hole collision (15 points)

What is the minimum mass M_2 of a Schwarzschild black hole that results from the collision of two Kerr black holes of equal mass M_1 and opposite angular momentum parameter, a? From which fundamental principle can the second law of black hole thermodynamics be derived?

Hint: Remember that according to the second law of black hole thermodynamics the total black hole surface area cannot decrease in any physical process. You can *derive the surface element on the horizon of a black hole from its metric.*

Problem 3: The gravity of light (35 points)

In this exercise we consider how propagating photons can curve spacetime, resulting in the so-called pp-wave backgrounds. In general, a pp-wave in four dimensions is given by the metric:

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + \Phi(u,x,y)\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{3}$$

where u, v are light-cone coordinates $u \equiv z - t, v \equiv z + t$, and x and y are the remaining usual Cartesian coordinates.

1. Write down the Einstein equations in vacuum using (3) as an ansatz. Show that a solution for Φ is

   $$\Phi_0 = \kappa_+(u)(x^2 - y^2) + 2\kappa_\times(u)xy\ .$$

   Show that if $\kappa_+(u)$ and $\kappa_\times(u)$ are finite, then the Riemann tensor for this solution is finite in all spacetime and the Ricci tensor and the scalar curvature vanish.

2. If the amplitudes $\kappa_+(u), \kappa_\times(u)$ are well-behaved, the above solution represents a non-singular gravitational wave spacetime. Convince yourself that these modes have a spin-2 behavior.

   Hint: Think about a rotation around the z-axis represented by coordinate transformation:

   $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\ .$$

   Consider how the amplitudes $\kappa_+(u), \kappa_\times(u)$ rotate in order to keep Φ_0 invariant.

3. Now let us include a standard Maxwell field with an energy-momentum tensor

   $$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\ . \tag{4}$$

For waves traveling in the (originally) z-direction, the field strength is given by

$$F = 2F_{ui}\mathrm{d}u \wedge \mathrm{d}x^i = 2\partial_u A_i(u)\mathrm{d}u \wedge \mathrm{d}x^i \ ,$$

such that the electric and magnetic fields are transverse,

$$E_i = -\varepsilon_{ij} B_j = F_{ui}(u) \ .$$

How do the Einstein equations change? Show that

$$\Phi = \Phi_0 + 4\pi G(x^2 + y^2)(\vec{E}^2 + \vec{B}^2)$$

is a solution. What is the value of the Ricci scalar in this case?

Problem 4: Conformal diagram (15 points)

Draw the conformal (Penrose-Carter) diagram of Minkowski spacetime and label it as it is standardly done. Explain the meaning of any special hypersurfaces and indicate the orientation of lightcones.