GENERAL RELATIVITY MIDTERM

08.11.2013.

Please write your solutions to each of the four problems on a separate sheet of paper! You have 3 hours. Good luck!

Some formulas:

- Christoffel symbol: $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})$,
- Riemann tensor: $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma}$.

■ PROBLEM 1 Theoretical questions. *(8 points)*

Answer the following briefly (one or two sentences).

(a) Define a manifold of class C^p, where p is a positive integer.

(b) Define a Cauchy surface of an n-dimensional manifold $\mathcal{M}$.

(c) Explain the difference between the Strong Equivalence Principle and the Einstein Equivalence Principle.

(d) What is the necessary and sufficient condition for vanishing of torsion on a manifold?

■ PROBLEM 2 Geodesics in Rindler space-time. *(9 points)*

The invariant element of Rindler space-time is given by

$$ds^2 = dw^2 - \left(1 + \frac{gw}{c^2}\right)^2 (dw^0)^2 \tag{2.1}$$

(a) *(1 pt.)* What is the velocity of light in these coordinates ($c\,dw/dw^0$)?

(b) *(2 pts.)* Write down all the geodesic equations for the motion of a particle in Rindler space-time.

(c) *(2 pts.)* Combine the equations from (b) into one equation of motion for $w(w^0)$.

(d) *(2 pts.)* Show that the solution for a particle starting at $w = \overline{w}$ at time $w^0 = 0$ with velocity zero is

$$w\left(w^0\right) = \frac{c^2}{g}\left[\left(1 + \frac{g\overline{w}}{c^2}\right)\frac{1}{\cosh\left(\frac{gw^0}{c^2}\right)} - 1\right] . \tag{2.2}$$

(e) *(2 pts.)* Calculate the velocity of the particle as a function of w^0 ($v = c\,dw/dw^0$). Does the particle ever travel faster than light?

■ PROBLEM 3 Tensor identities. *(8 points)*

(a) *(5 pts.)* If $A^{\mu\nu}$ is a symmetric tensor, $A^{\mu\nu} = A^{\nu\mu}$, and $B^{\mu\nu}$ is an anti-symmetric tensor, $B^{\mu\nu} = -B^{\nu\mu}$, prove the following identities,

$$\text{(i)} \qquad [\nabla_\mu, \nabla_\nu] A^{\mu\nu} = 0 \,, \tag{3.1}$$

$$\text{(ii)} \qquad [\nabla_\mu, \nabla_\nu] B^{\mu\nu} = 0 \,, \tag{3.2}$$

where $[X, Y] = XY - YX$ is a commutator.

(b) *(3 pts.)* Show the following identity for an arbitray metric $g_{\mu\nu}$,

$$\Box\phi \equiv \nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \right) \,, \tag{3.3}$$

where $g = \det(g_{\mu\nu})$. Hint: Show first that

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\nu \sqrt{|g|} \,. \tag{3.4}$$

■ PROBLEM 4 The curvature of a two-dimensional torus. *(10 points)*

A 2-dimensional torus T^2 can be embedded into a 3-dimensional Euclidean flat space as follows. The embedding metric is

$$dS^2 = dX^2 + dY^2 + dZ^2 \,, \tag{4.1}$$

and the surface of the torus is then defined by the constraint (see Figure 1)

$$(R - a)^2 + Z^2 = b^2 \,, \qquad b < a \,, \qquad R^2 = X^2 + Y^2 \,, \tag{4.2}$$

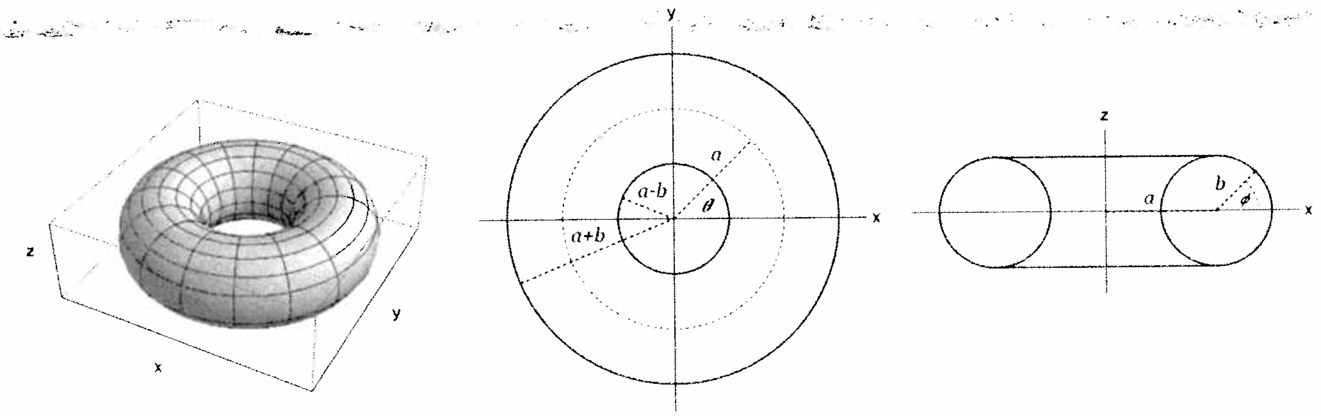

Figure 1: Left: A torus embedded into a 3-dimensional flat space. Middle: The $X-Y$ plane cross-section through $Z = 0$ of the torus. Right: The $Z-X$ planar cross-section of the torus through the $Y = 0$ plane.

(a) *(2 pts.)* Show that the angular coordinates $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$, defined by,

$$X = [a + b\cos(\phi)]\cos(\theta) \,, \qquad Y = [a + b\cos(\phi)]\sin(\theta) \,, \qquad Z = b\sin(\phi) \tag{4.3}$$

cover the torus completely, and that in these coordinates the metric is of the form,

$$ds^2 = [a + b\cos(\phi)]^2 d\theta^2 + b^2 d\phi^2 \,. \tag{4.4}$$

(b) *(3 pts.)* Calculate the nonvanishing connection coefficients , and show that the Ricci scalar equals

$$R = \frac{2\cos(\phi)}{b[a + b\cos(\phi)]}\,. \tag{4.5}$$

(c) *(3 pts.)* What is the minimum and what is the maximum value of the Ricci scalar (4.5)? Sketch the regions on the torus on which the Ricci scalar is positive, negative, and zero. Do your results agree with the naive expectation based on what you know about the Gauss' curvature, according to which the curvature of a two dimensional surface (manifold $\mathcal{M}$) at a point $p \in \mathcal{M}$ equals

$$\kappa = \frac{1}{r_1 r_2}\,, \tag{4.6}$$

where r_1 and r_2 are the principal radii of the ellipsoid (hyperboloid) constructed at p which describe local curvature at p. When both r_1 and r_2 are of an equal sign, one gets an ellipsoid, in which case the Gauss's curvature is positive, while when r_1 and r_2 are of an opposite sign one speaks of a hyperboloid (or a saddle), in which case the Gauss' curvature is negative.

(d) *(2 pts.)* Show that the *Euler characteristic* of any two-dimensional torus, defined by

$$\chi(T^2) = \frac{1}{4\pi}\int_{T^2} d^2x\sqrt{g}R \tag{4.7}$$

vanishes. That means that the average curvature of a two dimensional torus vanishes. This is true for any smooth two dimensional manifold of genus one (the genus g of a manifold $\mathcal{M}$ measures the number of holes on $\mathcal{M}$). Infact, the Euler characteristic depends only on the genus of the manifold. In two dimensions the formula is

$$\chi = 2(1-g)\,. \tag{4.8}$$

What are the Euler charachteristics of the surfaces of the two objects in Figure 2?

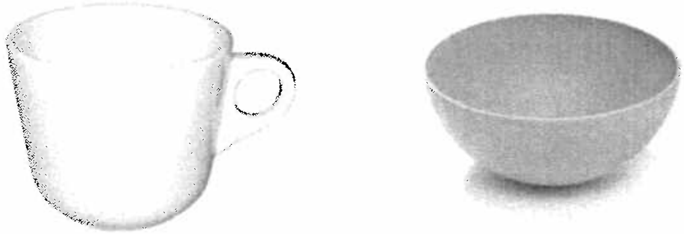

Figure 2: Left: a coffee cup; right: a bowl.