GENERAL RELATIVITY FINAL

31.01.2014.

Please write your solutions to each of the five problems on a separate sheet of paper! You have 3 hours. Good luck!

Some formulas:

- If K is a Killing vector, then $K_\mu \frac{dx^\mu}{d\lambda}$ is a conserved quantity along geodesics. In this exam we work in units in which $c = 1$.

(45 points total)

■ PROBLEM 1 Theoretical questions. *(8 pts.)*

Answer the following briefly (one or two sentences):

(a) *(2 pts.)* Define the Dominant Energy Condition (DEC).

(b) *(2 pts.)* Name all the maximally symmetric spaces. How many isometries do they have in D space-time dimensions?

(c) *(2 pts.)* Give a general definition of an event horizon.

(d) *(2 pts.)* How many dynamical degrees of freedom do gravitational perturbations (waves) have?

■ PROBLEM 2 The Reissner-Nordström black hole. *(12 pts.)*

The line element of the space-time of electrically charged non-rotating (Reissner-Nordström) black hole is

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2 \ , \tag{2.1}$$

where

$$B(r) = 1 - \frac{2MG_N}{r} + \frac{Q^2 G_N}{r^2} \ , \qquad Q^2 < M^2 G_N \ , \qquad d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2 \ . \tag{2.2}$$

(a) *(2 pts.)* Exploit the fact that the metric does not depend on coordinates t and φ to define two Killing vectors, associated with the time-translation symmetry and rotational symmetry, respectively. Write down the two conserved quantities along geodesics associated with these two Killing vectors.

(b) *(3 pts.)* Using the conserved quantities found in (a) write down the radial equation of motion for the motion in the equatorial plane ($\vartheta = \frac{\pi}{2}$) of this space-time for (i) a massive particle, and (ii) a massless particle, and define the respective effective potentials.

(c) *(4 pts.)* Find the circular orbits in the equatorial plane for the case of massles particles (find both of them!). Determine whether these orbits are stable or unstable with respect to perturbations in the radial direction. Where are these orbits with respect to the event horizons $r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - G_N Q^2}$?

(d) *(3 pts.)* A stationary observer sitting at fixed coordinates $(r_*, \vartheta_* = \frac{\pi}{2}, \varphi_*)$ outside of the outer horizon launches a projectile of a vanishing charge in the radial direction away from the black hole. What is the minimum velocity with which the projectile has to be launched so that it would escape to infinity? This is the so-called *escape velocity*.

■ PROBLEM 3 A conformal diagram. *(8 pts.)*

The de Sitter space-time line element in *global coordinates* is given by

$$ds^2 = -dt^2 + \frac{1}{H^2}\cosh^2(Ht)\left[d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right], \tag{3.1}$$

where $H = \text{const.}$ is the Hubble constant, and where the ranges of coordinates are

$$t \in \langle -\infty, \infty \rangle\,, \qquad \chi \in [0,\pi]\,, \qquad \vartheta \in [0,\pi]\,, \qquad \varphi \in [0, 2\pi\rangle\,. \tag{3.2}$$

Find the appropriate coordinate transformation (and write down the line elemnt explicitly in new coordinates), and draw and properly label the Carter-Penrose diagram for this space-time.

■ PROBLEM 4 The Kerr black hole. *(7 pts.)*

The line element of the Kerr black hole space-time in Boyer-Lindquist coordinates is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2MG_N r}{\rho^2}\right)dt^2 - \frac{4MG_N ar\sin^2\vartheta}{\rho^2}dt d\varphi \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2 + \frac{\sin^2\vartheta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\vartheta\right]d\varphi^2\,, \end{aligned} \tag{4.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\vartheta\,, \qquad \Delta = r^2 - 2MG_N r + a^2\,, \tag{4.2}$$

and where $|a| < MG_N$.

(a) *(3 pts.)* Determine the region of this space-time where it is impossible for an observer to sit still at some fixed spatial coordinates. Express the boundaries (both of them!) of this region in the form $r(\vartheta)$.

(b) *(4 pts.)* Use the Komar integral to calculate the total angular momentum J of the Kerr black hole. Guided by your answer, give the physical interpretation of the parameter a in the metric (4.1).

■ **PROBLEM 5** The Schwarzschild black hole. *(10 pts.)*

The line element of a Schwarzschild black hole space-time in Schwarzschold coordinates is given by

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) , \qquad A(r) = 1 - \frac{R_S}{r} , \tag{5.1}$$

where $R_S = 2MG_N$ is the Schwarzschild radius. Consider the motion of a massive particle orbiting this black hole in the equatorial plane ($\vartheta = \frac{\pi}{2}$).

(a) *(3 pts.)* Use the fact that

$$E = -K_\mu \frac{dx^\mu}{d\tau} \qquad \text{and} \qquad L = R_\mu \frac{dx^\mu}{d\tau} \tag{5.2}$$

are two conserved quantities along geodesics, where the Killing vectors are $K^\mu = (1, 0, 0, 0)$ and $R^\mu = (0, 0, 0, 1)$, in order to write down the radial equation of motion for the particle.

By analyzing the corresponding effective potential determine what is the condition for the existence of bounded orbits (in the radial direction), i.e. find the critical L. Sketch the effective potential and denote the region in which these bounded orbits lie.

(b) *(3 pts.)* Determine the radius of the stable circular orbit r_0 in terms of L, and the parameters of the space-time (5.1). What is the period of this orbit (i) in terms of the proper time of the particle, and (ii) in terms of the coordinate time (which is the proper time of a distant observer)? Express these periods in terms of the orbit radius r_0, L, and other parameters.

(c) *(4 pts.)* Assuming that the orbit of the particle is **close** to the circular orbit from (b) with the same L, determine the period of oscillations in the radial direction around the circular orbit (with respect to the proper time of the orbiting particle). What is this period in coordinate time? Express your answers in terms of the radius of the circular orbit, L, and other parameters.