GENERAL RELATIVITY RETAKE

14.03.2014.

(90 points in total)

Please write your solutions to each of the three problems on a separate sheet of paper! You have 3 hours. This written exam will be followed by an oral examination. Good luck!

A useful formula: $\int \frac{dx}{1+x^2} = \arctan(x)$.

■ PROBLEM 1 Geodesics. *(30 pts.)*

You are given a line element of a two dimensional space,

$$d\ell^2 = \frac{r^2}{r^2+a^2}dr^2 + (r^2+a^2)d\phi^2 \,, \tag{1.1}$$

where the ranges of coordinates are $r \in [0,\infty)$, $\phi \in [0, 2\pi)$.

(a) *(6 pts.)* Show that this is actually a 2-dimensional (flat) Euclidean space by explicitly constructing the required coordinate transformation, *i.e.* find $r(x,y)$ and $\phi(x,y)$ for which the line element reduces to the appropriate one. What is the Riemann tensor for this space?

(b) *(6 pts.)* Calculate the Christoffel symbols for this space in given coordinates (r,ϕ) and write down the geodesic equations.

(c) *(6 pts.)* By examining the line element (1.1) determine what the Killing vector for this space is and write down the conserved quantity associated with it. Denote this quantity by L.

(d) *(8 pts.)* Using the line element ℓ as the affine parameter calculate what the geodesics are in parametrized form, *i.e.* find what are $\phi(\ell)$ and $r(\ell)$ for $L \neq 0$.

Hint: use the conserved quantity and the line element.

Find the implicit relation $r = r(\phi)$ for the geodesics, *i.e.* show that

$$\frac{r^2+a^2}{L^2} - 1 = \tan^2(\phi - \phi_0)\,, \tag{1.2}$$

where $\phi_0 = \phi(\ell = 0)$.

(e) *(4 pts.)* Make use of the flat Euclidean coordinates you found in (a) to derive relation (1.2) in a simpler way.

■ **PROBLEM 2 Schwarzschild black holes in Kruskal-Szekeres coordinates.** *(30 pts.)*

In this problem you will show that the singularity at Schwarzschild radius, $R_S = 2MG_N/c^2$, of the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) c^2 dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 , \tag{2.1}$$

can be removed by making use of different coordinates.

We start by defining the *Kruskal-Szekeres* coordinates,
for $r > R_S$:

$$R = \left(\frac{r}{R_S} - 1\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \cosh\left(\frac{t}{2R_S}\right) , \tag{2.2}$$

$$T = \left(\frac{r}{R_S} - 1\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \sinh\left(\frac{t}{2R_S}\right) , \tag{2.3}$$

and for $r < R_S$:

$$R = \left(1 - \frac{r}{R_S}\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \sinh\left(\frac{t}{2R_S}\right) , \tag{2.4}$$

$$T = \left(1 - \frac{r}{R_S}\right)^{1/2} \exp\left(\frac{r}{2R_S}\right) \cosh\left(\frac{t}{2R_S}\right) . \tag{2.5}$$

(a) Show that the metric in Kruskal-Szekeres coordinates takes the following form

$$ds^2 = \frac{4R_S^3}{r} e^{-\frac{r}{R_S}} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2 . \tag{2.6}$$

(b) How is the radial null line parametrized in terms of R and T?

In order to get a better understanding of this space-time, sketch conformal diagram of the (maximally extended) Schwarzschild black hole in Kruskal-Szekeres coordinates and denote the regions that are separated by event horizons as I to IV. Denote on the diagram the singularities $r = 0$, the event horizons, and some lines of constant r and constant t (where r and t are the Schwarzschild radial and time coordinates). Explain the meaning of each of the regions. According to your understanding, which regions are certainly physical, and which are hypothetical? (Hypothetical is to be understood as potentially physical, but possibly not.) Recall that spherical coordinates are suppressed in this diagram such that every point represents a sphere S^2.

(c) What is the shape of the lines of constant r, for $r > R_S$? And for $r < R_S$? Can a massive object stay at a constant r inside the horizon?

(d) Draw a few lines of constant t. What happens for $t \to \infty$?

(e) Draw a path of an in-falling observer.

(f) Which regions are connected to the asymptotically flat region on the right part of the diagram (usually denoted as region I) by light signals?

■ **PROBLEM 3 Gravitational waves from a binary system.** *(30 pts.)*

Consider a gravitationally bound system of two stellar objects (stars, planets) of a mass M_1 and $M_2 \neq M_1$ rotating around each other at a distance r (from each other). The leading order amplitude of gravitational waves is described by the Einstein quadrupole formula,

$$h_{ij}^{TT}(\vec{x},t) = \frac{4G_N}{c^4}\int \frac{d^3x' T_{ij}^{TT}(\vec{x}\,',t_{\rm ret})}{\|\vec{x}-\vec{x}\,'\|}\,, \qquad \left(t_{\rm ret} = t - \frac{\|\vec{x}-\vec{x}\,'\|}{c}\right), \tag{3.1}$$

where G_N and c denote the Newton constant and the speed of light, respectively, and h_{ij}^{TT} is the graviton in physical gauge, *i.e.* h_{ij}^{TT} represents traceless and transverse gravitational waves ($\partial_i h_{ij}^{TT} = 0 = h_{ii}^{TT}$) and T_{ij}^{TT} are the transverse, traceless components of the stress-energy tensor. You may assume that the system is nonrelativistic, and work in the weak field approximation, in which the quadrupole formula (3.1) is valid.

(a) *(6 pts.)* By assuming that stars can be considered as point particles, show firstly that for this system and with a convenient choice of Cartesian coordinates ρ can be written in terms of the distance $r = r_1 + r_2$ between the stars, M_1, M_2, G_N and c as follows,

$$\rho(\vec{x},t) = M_1c^2\delta(x-r_1\cos(\omega t))\delta(y-r_1\sin(\omega t))\delta(z) + M_2c^2\delta(x+r_2\cos(\omega t))\delta(y+r_2\sin(\omega t))\delta(z)\,, \tag{3.2}$$

where $r_1 = (M_2 r)/(M_1+M_2)$, $r_2 = (M_1 r)/(M_1+M_2)$ and $\omega = \sqrt{G_N(M_1+M_2)}/r^{3/2}$.

(b) *(6 pts.)* Show that in radiation zone the quadrupole formula (3.1) can be simplified to,

$$\begin{aligned} h_{ij}^{TT}(\vec{x},t) &= \frac{2G_N}{c^6\|\vec{x}\,\|}\frac{d^2}{dt^2}J_{ij}(\bar{t}_{\rm ret}) \\ J_{ij}(\bar{t}_{\rm ret}) &= \int d^3x'\Big(x^{i\prime}x^{j\prime} - \frac{1}{3}\|\vec{x}\,'\|^2\delta_{ij}\Big)T_{00}(\vec{x}\,',\bar{t}_{\rm ret})\,, \qquad \left(\bar{t}_{\rm ret} = t - \frac{\|\vec{x}\|}{c}\right), \end{aligned} \tag{3.3}$$

where $T_{00} = \rho$ is the energy density.

(c) *(6 pts.)* Calculate the amplitude of gravitational waves in the radiation zone (3.3) for the system above for which ρ is given in (3.2), and show that it can be written as

$$h_{ij}^{TT}(\vec{x},t) = -\frac{4G_N\mu r^2\omega^2}{\|\vec{x}\,\|c^4}\begin{pmatrix} \cos(2\omega\bar{t}_{\rm ret}) & \sin(2\omega\bar{t}_{\rm ret}) & 0 \\ \sin(2\omega\bar{t}_{\rm ret}) & -\cos(2\omega\bar{t}_{\rm ret}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{t}_{\rm ret} = t - \frac{\|\vec{x}\|}{c}, \mu = \frac{M_1M_2}{M_1+M_2} \tag{3.4}$$

Give an argument to why gravitational waves are made up of spin $2\hbar$ particles!

(d) *(6 pts.)* The power dissipated by gravitation waves can be approximated by,

$$P_{\rm gw} = \frac{G_N}{5c^9}\left\langle \frac{d^3J_{ij}(\bar{t}_{\rm ret})}{dt^3}\frac{d^3J_{ij}(\bar{t}_{\rm ret})}{dt^3}\right\rangle, \tag{3.5}$$

where $\langle\cdot\rangle$ denotes a time average. Calculate the power emitted by the binary system at hand, and show that it can be written as,

$$P_{\rm gw} = \frac{32}{5c^5}\frac{G_N^4\mu^2(M_1+M_2)^3}{r^5}\,. \tag{3.6}$$

(e) *(6 pts.)* The gravitational power is radiated at the expense of orbital energy. Show that the separation between the stars depends on time as

$$r(t) \simeq \left[\frac{256G_N^3}{5c^5}(M_1+M_2)^2\mu(t_0-t)\right]^{1/4}, \tag{3.7}$$

if we assume the energy loss to be slow enough that the approximation of instantaneously circular orbits holds throughout. Here t_0 is the time in the future when the separation distance goes to zero. Of course, the weak field approximation, which was used to derive Eq. (3.7), will fail to be accurate when higher order gravitational corrections become important, which is well before $t = t_0$. Find t_0 in terms of the masses and the initial separation distance r_0.

Hint: In deriving (3.7) show first that the (Newtonian) energy of the system can be written as $E = -G_N \mu (M_1 + M_2)/(2r)$ which, in the absence of gravitational waves and other higher order relativistic effects, would be conserved.