GENERAL RELATIVITY FINAL

03.02.2017, Ruppert Blauw, 13:30-16:30

Please write your solutions to each of the five problems on a separate sheet of paper and write your name and student number on each sheet! You have 3 hours. Good luck!

A formula: $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$

- In this exam we work in units in which $c = 1$. The exam counts as 45% of the grade and contains in total 45 points. In this exam, G_N denotes Newton's constant.

X ■ PROBLEM 1 Theoretical questions. *(8 pts.)*

Answer the following briefly (one or two sentences):

(a) *(2 pts.)* Define event horizon.

(b) *(2 pts.)* Define trapped surface.

(c) *(2 pts.)* What is the ergoregion (in the book it is called ergosphere) of a rotating black hole?

(d) *(2 pts.)* Gravitational waves carry energy. State one observation based on which one can conclude that.

X ■ PROBLEM 2 Birkhoff-Jebsen theorem. *(15 points)*

By performing suitable coordinate transformations on a general metric and by making use of the spherical symmetry one can show that the general metric tensor corresponding to a spherically symmetric geometry (induced by a spherically symmetric mass distribution) can be written as the line element,

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\mu(t,r)}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{2.1}$$

where ν and μ are some functions of t and r, $\varphi \in [0, 2\pi)$ and $\vartheta \in [0, \pi]$ are spherical angles and r and t are the radial and time coordinates, respectively. Assume that you know the non-vanishing components of the Riemann tensor for the metric (2.1). They are,

$$R^0{}_{101} = \frac{1}{4}e^{\mu-\nu}\left[2\partial_t^2\mu + (\partial_t\mu)^2 - \partial_t\nu\partial_t\mu\right] + \frac{1}{4}\left[\partial_r\nu\partial_r\mu - 2\partial_r^2\nu - (\partial_r\nu)^2\right] .$$

$$R^0{}_{202} = -\frac{1}{2}re^{-\mu}\partial_r\nu , \qquad R^0{}_{303} = -\frac{1}{2}re^{-\mu}\sin^2\vartheta\,\partial_r\nu ,$$

$$R^0{}_{212} = -\frac{1}{2}re^{-\nu}\partial_t\mu , \qquad R^0{}_{313} = -\frac{1}{2}re^{-\nu}\sin^2\vartheta\,\partial_t\mu .$$

$$R^1{}_{212} = \frac{1}{2}re^{-\mu}\partial_r\mu , \qquad R^1{}_{313} = \frac{1}{2}re^{-\mu}\sin^2\vartheta\,\partial_r\mu , \qquad R^2{}_{323} = (1 - e^{-\mu})\sin^2\vartheta . \tag{2.2}$$

(a) *(4 points)* By making use of (2.2) calculate the non-vanishing components of the Ricci tensor. Show that these are, R_{00}, R_{11}, R_{22}, R_{33}, and $R_{01} = R_{10}$. When solving the Einstein vacuum equation you do not need to calculate the Ricci scalar. Why?

(b) *(2 points)* Show that the tr component of the Einstein vacuum equation implies that μ is independent of time, $\mu(t, r) = \mu(r)$.

(c) *(2 points)* Use (a linear combination of) the remaining components of the Einstein vacuum equation to show that

$$\nu(t, r) = -\mu(r) + f(t) . \tag{2.3}$$

(d) *(2 points)* Use these results to conclude that the most general spherically symmetric solution to the Einstein equation in the vacuum (after a suitable redefinition of the time coordinate) can be written as

$$ds^2 = -e^{-\mu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) . \tag{2.4}$$

(e) *(2 points)* Show that the other components of the Einstein vacuum equation imply,

$$e^{-\mu(r)} = 1 - \frac{R_S}{r} , \tag{2.5}$$

where R_S is a real constant, so that we obtain the Schwarzschild solution.

(f) *(3 points)* Make use of the Komar integral for the metric (2.4–2.5) to determine the physical meaning of R_S. Can R_S be negative? If yes, explain why yes; if not, explain why not.

Hint: Recall that the Komar integral reads,

$$E_R = \frac{1}{4\pi G_N}\int_{\partial\Sigma} d^2x\sqrt{\gamma_{\partial\Sigma}}n_\mu s_\nu \nabla^\mu K^\nu . \tag{2.6}$$

■ PROBLEM 3 Schwarzschild-de Sitter metric *(12 points)*

In this problem we shall consider the Schwarzschild-de Sitter space-time, whose metric (in static coordinates) is given by,

$$ds^2 = -\left(1 - \frac{2G_N M}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2G_N M}{r} - \frac{\Lambda}{3}r^2} + r^2 d\Omega^2 , \tag{3.1}$$

where M is the mass of a body, Λ is the cosmological constant, $d\Omega^2 = d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$ and $\varphi \in [0, 2\pi)$ and $\vartheta \in [0, \pi]$ are spherical angles.

(a) *(1 point)* Name the Killing vectors of the metric (3.1).

(b) *(2 points)* Consider a test particle in equatorial plane and show that the equation of motion can be written as,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r, E, L) = \frac{E^2}{2} , \tag{3.2}$$

where

$$V(r, E, L) = \frac{L^2}{2r^2} - \epsilon\frac{G_N M}{r} - \epsilon\frac{\Lambda}{6}r^2 - \frac{L^2 G_N M}{r^3} + \frac{\epsilon}{2} - \frac{L^2\Lambda}{6} . \tag{3.3}$$

Here λ is an affine parameter, $\epsilon = 1$ for massive particles and $\epsilon = 0$ for light and massless particles. $L = R_\mu dx^\mu/d\lambda$ is the angular momentum per unit mass. $E = -K_\mu dx^\mu/d\lambda$ is the energy of the test particle per its unit mass and R_μ and K_μ are the Killing vectors for rotations in the equatorial plane and for time translations, respectively.

(c) *(2 points)* Sketch $V(r)$ defined in Eq. (3.3) for $\epsilon = 1$ for the relevant qualitatively different cases (there are 3 distinct cases) (assume $L > 0$). Based on your drawings, what do you think, is there always a stable circular orbit in this metric?

(d) *(2 points)* Consider radial motion ($L = 0$) and show that there is a point in which, if the test particle is at rest, it can stay there forever. Calculate the radius of that point. Is that point stable under small perturbations, *i.e.* if radius increases (decreases) by a small amount, what will happen to the particle?

(e) *(3 points)* Show that the event horizons are determined by the equation.

$$1 - \frac{2G_N M}{r} - \frac{\Lambda}{3}r^2 = 0\,. \tag{3.4}$$

This equation has two real positive solutions. One represents the black hole event horizon r_{BH} and the other the cosmological event horizon r_c. The metric makes sense only if $r_{BH} < r_c$ and only in the region where $r_{BH} < r < r_c$. Assume for simplicity that the Hubble radius $r_H = \sqrt{3/\Lambda} \gg r_S \equiv 2G_N M$, and show that the approximate radii of the two event horizons are,

$$r_c \simeq r_H - \frac{r_S}{2}\,, \qquad r_{BH} \simeq r_S + \frac{r_S^3}{r_H^2}\,. \tag{3.5}$$

Hint: To find (3.5) expand around $r = r_H$ for the first solution and around $r = r_S$ for the second solution.

(f) *(2 points)* Consider a test particle whose energy $E = 1$ and show that its coordinate velocity, defined by dr/dt ($t \equiv x^0$) at the black hole event horizon vanishes. Can you interpret that result?

■ PROBLEM 4 Energy in Schwarzschild-de Sitter spacetime *(10 points)*

We have seen two possible definitions of the total energy for spacetimes that admit a timelike Killing vector:

$$E_T = \int_\Sigma d^3x\sqrt{\gamma}n_\mu J_T^\mu\,, \qquad E_R = \frac{1}{4\pi G_N}\int_\Sigma d^3x\sqrt{\gamma}n_\mu J_R^\mu\,. \tag{4.1}$$

where γ_{ij} is the induced metric on a spacelike hypersurface Σ and n_μ is the unit normal vector associated with Σ (when Σ is spacelike, then n_μ is timelike). Here the conserved currents are defined as

$$J_T^\mu = K_\nu T^{\mu\nu}; \quad J_R^\mu = K_\nu R^{\mu\nu}, \tag{4.2}$$

for Killing vector K^μ associated with time translations. In this problem you should interpret the cosmological constant Λ as the vacuum energy contribution to the energy-momentum tensor $T^{\mu\nu}$, *i.e.* Λ contributes as, $T^{\mu\nu} \propto \Lambda g_{\mu\nu}$. In this problem you can use results from problem 3.

(a) *(1 point)* In de Sitter-Schwarzschild in the given coordinates, what are the meaningful spacelike (spherical) hypersurfaces to consider?

(b) *(2 points)* Use the Killing vector identity, $\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$ (you do not need to prove this identity) and Stokes' theorem to express E_R as a two-dimensional integral.

(c) *(2 points)* A priori, it is not clear that a similar trick can be used for E_T. However, using for instance the definition of the Einstein-Hilbert action including a cosmological constant,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} (R - 2\Lambda) \right], \tag{4.3}$$

one can find a relation between J^μ_T and J^μ_R. Derive this relation.

Hint: That relation can be obtained by deriving a relation between $T_{\mu\nu}$ and $R_{\mu\nu}$.

(d) *(3 points)* Compute E_T and E_R as a function of the radius r of $\partial\Sigma$. Are E_T and E_R equal? For simplicity, for $\partial\Sigma$ choose a sphere of constant radius.

(e) *(2 points)* The energy-momentum tensor of the 'vacuum energy' Λ is of the perfect fluid form, $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$, for timelike four-velocity $U^\mu = (U^0, 0, 0, 0)$. What are ρ and p in terms of Λ? Express the Λ-dependent contribution to E_T and E_R in terms of ρ and p. Can you interpret them physically?