GENERAL RELATIVITY MIDTERM EXAM

10 Nov 2017, 13:30 in Ruppert Blauw

Please write your solutions to each of the four problems on a separate sheet of paper! This is a **closed book** exam. You have **3 hours**. In total 35 points=35% of total grade. Good luck!

Some useful formulas:

- Killing vector identities: $\nabla_\mu \nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma} K^\sigma$, $K^\alpha \nabla_\alpha R = 0$.
- Riemann tensor: $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)$.

■ PROBLEM 1 Theoretical questions *(8 points)*

Answer the following briefly (one or two sentences).

(a) *(2 points)* Define a local inertical frame (LIF).

(b) *(2 points)* When is the *Cauchy surface* in general relativity defined and what does that imply for the initial value problem?

(c) *(2 points)* Imagine that you are an accelerated observed and that you measure redshifted light from an observer at rest. How would you use this fact to argue that gravitational redshift must exist?

(d) *(2 points)* List all the independent symmetries of the Riemann tensor.

■ PROBLEM 2 Light Cones *(5 points)*

Consider a 1+1 dimensional space-time, whose metric $(0,2)-$tensor is given by,

$$ds^2 = -dt \otimes dt + a(t)^2 dx \otimes dx \,. \tag{2.1}$$

where $a(t)$ is a scale factor ($a(t)dx$ is a 1-form field that can be used to measure physical distances and dt is the 1-form field that can be used to measure time lapses). Assume that

$$a(t) \;=\; t^{1/\epsilon} \quad (t > 0)\,. \tag{2.2}$$

The constant ϵ is defined as the rate of change of the inverse expansion rate: $\epsilon = (d/dt)[1/H(t)]$, where $H(t) = (d/dt)\ln[a(t)]$. Consider a vector field, $V = d/d\lambda$ ($V^\mu = dx^\mu/d\lambda$ in some coordinate system x^μ). The light cones are then defined as

$$ds^2(V,V) = 0\,. \tag{2.3}$$

(a) *(1 point)* Show that (2.3) implies the following differential equations for the light-cones,

$$\frac{dt}{d\lambda} = \pm a(t)\frac{dx}{d\lambda}\,. \tag{2.4}$$

(b) *(4 points)* Solve this equation for expanding space-times (whose scale factor is given by Eq. (2.2)) for $\epsilon \neq 1$ and $\epsilon \neq 0$. Show that the solution can be written as

$$x(t) = \pm \frac{t^{1-1/\epsilon}}{1-1/\epsilon} + x_0 \,. \tag{2.5}$$

Sketch how the light cones for some point $\infty > t > 0$ look like for $\epsilon > 1$ (decelerating expansion, $d^2a/dt^2 < 0$). In general relativity it may happen that light cones cannot intersect (in the past or in the future) or geodesics can stop. If you find any of these to be realized in this problem, *note* that and provide a *physical explanation* for what it means.

■ PROBLEM 3 An embedded surface *(12 points)*

In this problem, consider a two-dimensional surface $S \subset \Sigma$ embedded in Euclidean three-space Σ parameterized by cylindrical coordinates (r, θ, z). The embedding is defined by $\theta \mapsto \theta$, $r \mapsto r$, and $z \mapsto z(r)$, where the function $z(r)$ is arbitrary for now.

(a) *(1 point)* Starting from the line element in cylindrical coordinates, show that the induced metric on the embedded surface is given by

$$\mathrm{d}s^2 = \left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right] \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2.$$

(b) *(2 points)* Show that the nonvanishing Christoffel symbols are given by

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r}, \qquad \Gamma^{r}_{rr} = \frac{z'z''}{1+z'^2}, \qquad \Gamma^{r}_{\theta\theta} = \frac{-r}{1+z'^2},$$

where primes denote derivatives with respect to r.

(c) *(2 points)* Write down all components of the Killing equation, and verify that the vector $K = \partial_\theta$ is a Killing vector. What symmetry does this Killing vector correspond to?

(d) *(3 points)* Show that the Ricci scalar is equal to,

$$R = \frac{2z'z''}{r(1+z'^2)^2}.$$

We now impose that we want the Ricci scalar to be constant and nonnegative, $R = R_0$. You may use this in the remainder of the exercise.

(e) *(2 points)* The Ricci scalar being constant requires z to satisfy a differential equation. Show that this equation can be integrated once to yield

$$\frac{-1}{1+z'^2} = \frac{R_0 r^2}{2} + A.$$

Derive a condition for the integration constant A from the requirement that the embedding should be smooth for all $r < \sqrt{2/R_0}$, and show that

$$z' = \pm\sqrt{\frac{R_0 r^2}{2 - R_0 r^2}}.$$

(f) *(1 point)* Integrate the last equation to yield $z(r)$. Sketch the shapes of the embedded surfaces in the cases $R_0 > 0$ and $R_0 = 0$.

(g) *(1 point)* In two dimensions, does the requirement that the Ricci scalar be constant completely specify the geometry locally (in some small neighborhood), or is additional information required?

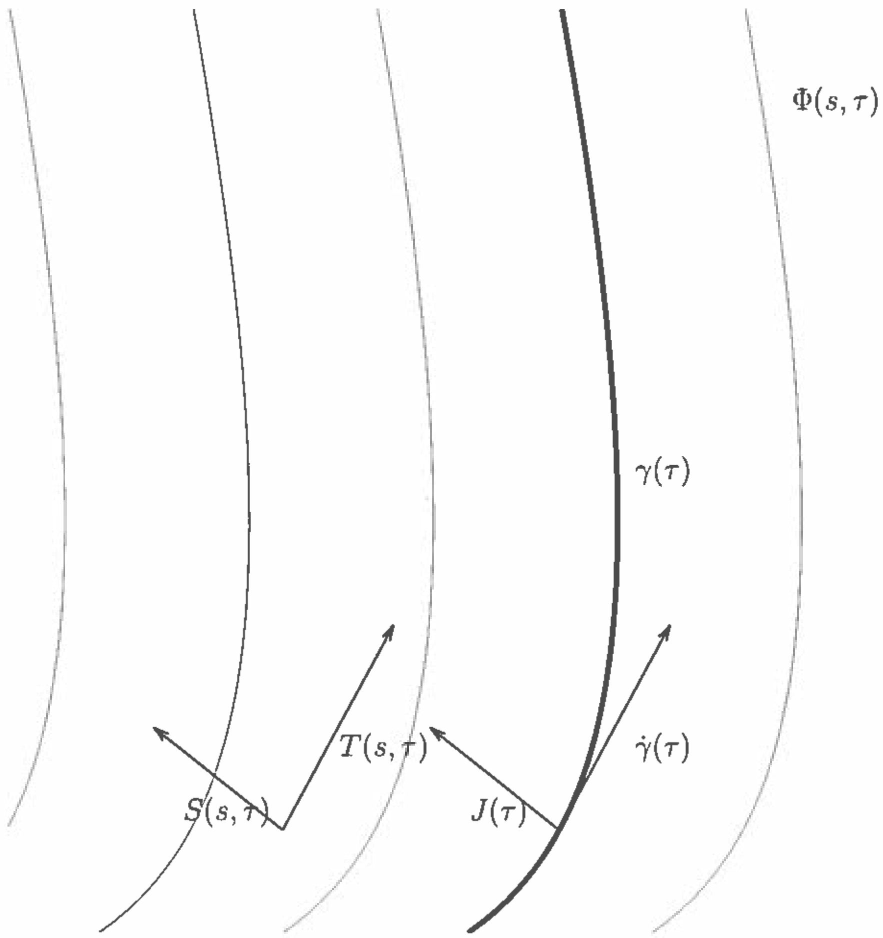

Figure 1: Here you see the congruence of geodesics $\Phi(s,\tau)$ on $\mathcal{M}$. The bold line is $\Phi(0,\tau) = \gamma(\tau)$. The other lines are $\Phi(s_0,\tau)$, so they are also geodesic segments. You see the direction of $T(s,\tau) = \partial_\tau\Phi(s,\tau)$ and $S(s,\tau) = \partial_s\Phi(s,\tau)$. The variation field is $J(\tau) = S(0,\tau)$.

■ PROBLEM 4 Jacobi fields *(10 points)*

In this exercise we will study congruences of geodesics in a manifold $\mathcal{M}$. Let $\gamma : [\tau_\mathrm{i}, \tau_\mathrm{f}] \to \mathcal{M}$ be a segment of a specific geodesic. We define a congruence of geodesics by a function $\Phi : (-\epsilon, \epsilon) \times [\tau_\mathrm{i}, \tau_\mathrm{f}] \to \mathcal{M}$, such that $\Phi^\mu(0, \tau) = \gamma^\mu(\tau)$ and every curve $\Phi(s_0, \tau)$ is a segment of a geodesic, see figure 1.

The vector field $J^\mu(\tau) = \partial_s \Phi^\mu(s, \tau)|_{s=0}$ is the variation field of Φ.

(a) *(3 points)* Show that for a vector field V^μ

$$S^\lambda \nabla_\lambda \left(T^\rho \nabla_\rho V^\mu\right) - T^\rho \nabla_\rho \left(S^\lambda \nabla_\lambda V^\mu\right) = R^\mu{}_{\sigma\lambda\rho} S^\lambda T^\rho V^\sigma, \tag{4.1}$$

where $T^\mu(s, \tau) = \partial_\tau \Phi^\mu(s, \tau)$ and $S^\mu(s, \tau) = \partial_s \Phi^\mu(s, \tau)$.
Hint: Recall the identity, $[\nabla_\lambda, \nabla_\rho] V^\mu = R^\mu{}_{\sigma\lambda\rho} V^\sigma$.

(b) *(3 points)* By making use of Eq. (4.1), show that J^μ satisfies the following equation

$$\dot\gamma^\rho \dot\gamma^\lambda \nabla_\rho \nabla_\lambda J^\mu = R^\mu{}_{\rho\sigma\lambda} J^\lambda \dot\gamma^\rho \dot\gamma^\sigma. \tag{4.2}$$

This equation is called the Jacobi equation. Solutions of Eq. (4.2) are called Jacobi fields. It can be shown that if $J^\mu(\tau)$ is a Jacobi field, there exists a congruence of geodesics Ψ with J^μ as variation field.

(c) *(2 points)* Show that $\dot\gamma$ and $\tau\dot\gamma$ are Jacobi fields.

(d) *(2 points)* Show that, when the manifold $\mathcal{M}$ has a Killing vector K^μ, this Killing vector field restricted to γ is a Jacobi field.