GENERAL RELATIVITY MIDTERM EXAM

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Please write your solutions to each of the four problems on a separate sheet of paper! This is a **closed book** exam. You have **3 hours**. In total 35 points =35% of the total grade (plus 4 bonus points). Good luck!

Some useful formulas:

- Killing vector identities: $\nabla_\mu \nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma} K^\sigma$, $K^\alpha \nabla_\alpha R = 0$.
- Riemann tensor: $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)$.

■ PROBLEM 1 Theoretical questions *(8 points)*

Answer the following briefly (one or two sentences).

(a) *(2 points)* Give arguments that show that Local Inertial Frames (LIFs) exist in general relativity.

(b) *(2 points)* Define the *future domain of dependence* $D^+(S)$ of an achronal subset S of spacetime manifold $\mathcal{M}$, $S \subset \mathcal{M}$.

(c) *(2 points)* When considering curved spaces, what is the reason to introduce a connection?

(d) *(2 points)* Define Einstein's Equivalence Principle. What does it tell us about how to generalize the special relativistic action for a scalar field $\phi(x)$, $S_{SR}[\phi] = \int d^4x \Big[-\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi) \Big]$, to its general relativistic counterpart? $V(\phi)$ is a scalar field potential and $\eta_{\mu\nu}$ is the metric tensor on flat Minkowski space.

■ PROBLEM 2 Geodesics on the Poincaré half plane *(9 points)*

Consider the spacetime metric

$$ds^2 = \frac{a^2}{y^2}\left(dx^2 + dy^2\right) \tag{2.1}$$

with $y > 0$.

(a) *(2 points)* Find all non-vanishing Christoffel symbols associated with the metric (2.1). *Hint:* They are given by: $\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}$, $\Gamma^y_{xx} = \frac{1}{y}$ and $\Gamma^y_{yy} = -\frac{1}{y}$.

(b) *(3 points)* Write down the geodesic equations and show that all geodesics must obey the equation $(x - x_0)^2 + y^2 = l^2$ for arbitrary constants x_0 and l.
Hint: First show that $P_x = \frac{a^2}{y^2}\frac{dx}{ds}$ is a conserved quantity. Then derive the following equation, $\frac{dy}{dx} = \pm\frac{a}{yP_x}\sqrt{1 - \frac{P_x}{a^2}y^2}$, which you can easily integrate.

(c) *(2 points)* Sketch a representative of each of the two distinct classes of geodesics in the upper-half plane.

(d) *(2 points)* Compute the proper length $\Delta s = \int ds$ along a line with fixed coordinate $x = $ const. What happens on approach to the half-plane boundary at $y = 0$?

■ PROBLEM 3 The energy of a point particle on cosmological spaces *(4 points)*

Consider a flat, 1+3 dimensional, cosmological metric of the form,

$$ds^2 = -c^2dt^2 + a^2(t)d\vec{x}\cdot d\vec{x} \tag{3.1}$$

where $d\vec{x}\cdot d\vec{x} = dx^2 + dy^2 + dz^2$ and $a = a(t)$ is the scale factor.

(a) *(2 points)* Show that in this metric, $p^2 \equiv g_{\mu\nu}p^\mu p^\nu = -m^2c^2$.

(b) *(2 points)* Derive the following Einstein's relation,

$P g_{\mu\nu} =$

$$E^2 = \frac{\vec{P}^2c^2}{a(t)^2} + m^2c^4\,, \tag{3.2}$$

where $\vec{P} = (P^1, P^2, P^3)^T$ is the conserved 3-momentum. Is energy conserved on expanding spaces? Give argument(s) that support your answer!
Hint: Construct the Killing vectors for spatial translations.

■ PROBLEM 4 Comparing the Poincaré plane and Hyperbolic space *(9 points)*

Consider the following two spaces:

$$ds^2 = \frac{a^2}{y^2}\left(dx^2 + dy^2\right) \tag{4.1}$$

$$ds^2 = a^2\left(d\chi^2 + \sinh^2\chi d\phi^2\right) \tag{4.2}$$

with $y > 0$.

(a) *(3 points)* Compute the curvature scalar and – based on your result – conclude whether (4.1) and (4.2) represent flat or curved spaces.

(b) *(3 points)* How many Killing vectors do they have? Construct all of them for the metric (4.1).

(c) *(2 points)* Are (4.1) and (4.2) maximally symmetric spaces? Explain why!

(d) *(1 points)* Are these two spaces (locally) equivalent?

■ PROBLEM 5 Scalar field theory *(5 points + 4* bonus points)*

Consider a real scalar field ϕ in four dimensional space-time whose action is given by,

$$S[\phi] = \int d^4x\sqrt{-g}\Big(-\frac{1}{2}F(\phi)g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\Big)\,. \tag{5.1}$$

where $V(\phi)$ is a scalar field potential and $F(\phi)$ is a kinetic function of ϕ, such that for a canonically normalized field, $F(\phi) = 1$, and $g = \det[g_{\mu\nu}]$ is the determinant of the metric tensor $g_{\mu\nu} = g_{\mu\nu}(x)$. Assume, for simplicity, that the metric tensor is diagonal, *i.e.* $g_{\mu\nu}(x) = \mathrm{diag}(g_{00}(x), g_{11}(x), g_{22}(x), g_{33}(x))$.

(a) *(2 points)* Use the variation principle, $\delta S/\delta\phi(x) = 0$, to show that the Euler-Lagrange equation for ϕ can be written as,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}F(\phi)g^{\mu\nu}\partial_\nu\phi(x)\right] - \frac{1}{2}\frac{dF(\phi)}{d\phi}g^{\mu\nu}(\partial_\mu\phi(x))(\partial_\nu\phi(x)) - \frac{dV(\phi)}{d\phi(x)} = 0\,. \tag{5.2}$$

Prove that this equation is generally covariant, *i.e.* that it is a scalar under general coordinate transformations.

(b) *(1 point)* Show that the canonical momentum of ϕ is,

$$\pi(x) = -\sqrt{-g}g^{00}F(\phi)\partial_0\phi(x)\,. \tag{5.3}$$

(c) *(2 points)* By making use of a Legendre transformation, construct the Hamiltonian for the scalar field whose action is (5.1).
Hint: Show that

$$H[\phi,\pi;x] = \int d^3x\mathcal{H}(\phi,\pi;x)\,. \tag{5.4}$$

where the Hamiltonian density $\mathcal{H}(\phi,\pi;x)$ is,

$$\mathcal{H}(\phi,\pi;x) = -\frac{1}{2}\frac{\pi^2}{\sqrt{-g}F(\phi)g^{00}} + \frac{1}{2}\sqrt{-g}F(\phi)g^{ij}(\partial_i\phi)(\partial_j\phi) + \sqrt{-g}V(\phi)\,, \tag{5.5}$$

where $i,j = 1,2,3$.

(d) *(2* bonus points)* Derive the Hamilton equations and show that they can be written as,

$$\frac{d\phi}{dt} = -\frac{\pi}{\sqrt{-g}F(\phi)g^{00}} \tag{5.6}$$

$$\frac{d\pi}{dt} = -\frac{dF}{d\phi}\left[\frac{\pi^2}{2\sqrt{-g}F^2(\phi)g^{00}} + \frac{1}{2}\sqrt{-g}g^{ij}(\partial_i\phi)(\partial_j\phi)\right] + \partial_i\left[\sqrt{-g}g^{ij}F(\phi)(\partial_j\phi)\right] - \sqrt{-g}\frac{dV(\phi)}{d\phi}\,. \tag{5.7}$$

Hint: You can either vary the action, $S[\phi,\pi] = \int d^4x\,[(\partial_0\phi)\pi - \mathcal{H}(\phi,\pi;x)]$, or use the Poisson brackets method.

(e) *(2* bonus points)* Show that the Hamilton equations (5.6–5.7) are equivalent to the Euler-Lagrange equations (5.2), and thus covariant.