GENERAL RELATIVITY MIDTERM EXAM

11 Nov 2022, 13:30 – 16:30, KBG Cosmos, Pangea, Atlas & BBG - 0.05 (extra time).

Please write your solutions to each of the four problems on a separate sheet of paper! This is a **closed book** exam. You have **3 hours**. In total 35 points (+3 bonus pts) = 35% of the total grade. *Please write each question on a separate sheet of paper and sign each sheet!* Good luck!

- Riemann tensor: $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)$.

■ PROBLEM 1 Theoretical questions *(8 points + 1 bonus point)*

Answer the following concisely.

(a) *(2 points)* Define a Killing vector field on an n-dimensional manifold $\mathcal{M}$.

(b) *(2 points)* Give a definition of a local inertial frame (LIF).

(c) *(2 points)* Define the future domain of dependence $D^+(S)$ of a closed achronal set $S \subset \mathcal{M}$, which is subset of a manifold $\mathcal{M}$.

(d) *(2 points + 1 bonus point)* Think of the coffee in a coffee cup as a closed, compact smooth manifold (even though coffee consists of molecules, assume that these can be smoothed out). Prove that after stirring the coffee, there is a point (a molecule) that has not moved by the stirring.

Hint: Use Hadamard-Brouwer's fixed point theorem [1910, 1911], which states that for any continuous map ϕ that maps a three-ball B^3 to itself, there exists a point $x_0 \in B^3$ that is a fixed point of the map, *i.e.* $\phi(x_0) = x_0$.
NB: The Brouwer's generalization uses a compact convex set in $n \in \mathbb{N}_0$ dimensions rather than B^3, but for this question the Hadamard's version suffices.

■ PROBLEM 2 Relativistic particle in an electromagnetic field. *(4 points)*

The action of a charged particle in the presence of an electromagnetic field is given by

$$S = -mc^2 \int_a^b \mathrm{d}\tau + \frac{e}{c}\int A_\mu(x^\nu)\mathrm{d}x^\mu, \qquad (2.1)$$

where $d\tau = \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}/c$ is the proper time, m denotes particle's mass and e is the electric charge of the particle that dictates its interaction strength with the vector potential A^μ.
It can be shown that the equations of motion of the particle, written in covariant form, are

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{e}{mc}F^{\mu\nu}p_\nu\,, \qquad (2.2)$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the electromagnetic covariant field strength.

(a) *(2 points)* Recalling that the components of the electric field E^i and those of the magnetic field B^i are related to the components of the field strength as $E^i = F^{0i}$, $F^{ij} = \epsilon^{ijk}B^k$, show that the covariant equations of motion (2.2) splits in components as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma mc^2) = e\vec{E}\cdot\vec{v}\,, \qquad \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\,. \qquad (2.3)$$

Explain the physical significance of the right hand sides of both equations.

Consider a particle that is subjected only to a constant magnetic field, that we assume to be oriented in the z-direction, namely $\vec{B} = B_0\hat{z}$, with $\partial_\mu B_0 = 0$. One can show that, in the laboratory frame, the particle describes a *helix*, with its coordinates $(x(t), y(t), z(t))$ evolving in time according to the following equations

$$\begin{cases} x(t) = \frac{v_{x,0}}{\Omega}\sin(\Omega t) - \frac{v_{y,0}}{\Omega}\cos(\Omega t) + x_0\,, & (2.4) \\ y(t) = \frac{v_{y,0}}{\Omega}\sin(\Omega t) + \frac{v_{x,0}}{\Omega}\cos(\Omega t) + y_0\,, & (2.5) \\ z(t) = v_{z,0}t + z_0, & (2.6) \end{cases}$$

where $v_{x,0} = v_x(0)$, $v_{y,0} = v_y(0)$ and $v_{z,0} = v_z(0)$ are initial velocities, $z_0 = z(0)$ and x_0, y_0 are related to the initial position of the particle. Moreover, $\Omega = \omega/\gamma$, $\omega = eB_0/(mc)$, with $B_0 = \|\vec{B}_0\|$ is the Larmor frequency, also known as *cyclotron frequency*. We can thus define a *laboratory period* T_L that is related to the period T_p measured in the frame of the particle as

$$T_L = \frac{2\pi}{\Omega} = T_p\gamma\,, \qquad \text{with} \quad T_p = \frac{2\pi}{\omega}\,. \tag{2.7}$$

(b) *(2 points)* Use these results to explain the twin paradox.

■ PROBLEM 3 Differential geometry on the two-sphere *(7 points)*

In this problem we will show two foundational results in differential geometry applied to the unit two-sphere S^2. Recall that the metric of the unit two-sphere can be written as

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\,, \tag{3.1}$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

For an n-dimensional Riemannian manifold $\mathcal{M}$ we define the *volume* $V(\mathcal{M})$ of $\mathcal{M}$ with respect to the Levi-Civita tensor as

$$V[\mathcal{M}] = \int_{\mathcal{M}} \epsilon = \int_{\mathcal{M}} \sqrt{|g|}\mathrm{d}^n x\,. \tag{3.2}$$

(a) *(2 points)* Using (3.2), compute the volume $V[S^2]$ of the unit two-sphere.

For a two-dimensional compact, orientable, Riemannian manifold $\mathcal{M}$ (without boundary), we define the *Euler characteristic* $\chi(\mathcal{M})$ as

$$\chi(\mathcal{M}) = \frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{|g|}R\,\mathrm{d}^2x\,, \tag{3.3}$$

where R denotes the Ricci scalar. According to the Gauss-Bonnet theorem, there is a relation between $\chi(\mathcal{M})$ and the *genus* g of $\mathcal{M}$ defined by

$$\chi(\mathcal{M}) = 2 - 2g\,. \tag{3.4}$$

Intuitively, g counts the number of 'holes' in M.

(b) *(2 points)* Using (3.3) and (3.4), compute the Euler characteristic of the unit two-sphere as well as its genus. According to your result, how many 'holes' does the two-sphere have?

For an n-dimensional orientable, Riemannian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ and a vector field V^μ on $\mathcal{M}$, Stokes' theorem states that

$$\int_{\mathcal{M}} \mathrm{d}^n x\sqrt{|g|}\nabla_\mu V^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1}y\sqrt{|\gamma|}n_\mu V^\mu\,, \tag{3.5}$$

where γ denotes the induced metric on $\partial\mathcal{M}$ and n^μ is the unit normal to the boundary.

(c) *(3 points)* Choose $\mathcal{M}$ to be the upper hemisphere of the two-sphere, *i.e.* the space obtained by restricting $\theta \in \left[0, \frac{\pi}{2}\right]$, and consider the vector field

$$V = \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\,. \tag{3.6}$$

Explicitly verify that Stokes' theorem holds in this case, by evaluating both sides of (3.5) independently.

■ PROBLEM 4 The Misner space *(16 points + 2 bonus points)*

The *Misner space* is a two-dimensional spacetime that allows for *closed timelike curves* (CTSs). Namely, the causal structure of the spacetime admits curves – for instance traced by the worldline (geodesic) of moving objects – that come back to their starting spacetime point. Said differently, the Misner space allows for 'time travel'. In the following, we will first show how the Misner space is constructed, and then we will examine its causal structure.

Construction of the Misner space. Analogously to Rindler space, the Misner spacetime can be covered by some portions of the Minkowski space as follows. Consider the two-dimensional Minkowski spacetime, with timelike coordinate t and spacelike coordinate x, with metric,

$$ds^2 = -dt^2 + dx^2\,, \tag{4.1}$$

in units in which $c = 1$. We subdivide the Minkowski spacetime in four regions bounded by the lines $t = \pm x$, as in Figure 1.

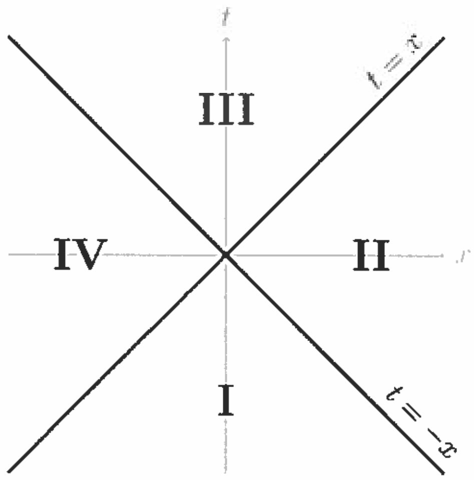

Figure 1

The Misner space is described by two coordinates, T and z, that are related to the Minkowkski space coordinates t and x *via* the following relations:

$$(1): \begin{cases} t = -2\sqrt{-T}\cosh z \\ x = -2\sqrt{-T}\sinh z \end{cases}, \qquad (2): \begin{cases} t = 2\sqrt{T}\sinh z \\ x = 2\sqrt{T}\cosh z \end{cases}, \tag{4.2}$$

with (1) defined for $T < 0$, $-\infty < z < \infty$ and (2) for $T > 0$, $-\infty < z < \infty$.

(a) *(2 points)* Varying T and z in their domains, one cannot span the full Minkowski spacetime. Which regions of Minkowski spacetime, among those depicted in Figure 1, are covered by the (T, z) coordinates defined by (1) and which by (2) in Eq. (4.2)?

(b) *(2 points)* In the regions identified in (a), sketch a couple of curves corresponding to constant T coordinate, as well as a couple of curves corresponding to constant z coordinate. Are these curves timelike or spacelike when viewed in the (t, x) coordinates?

(c) *(2 points)* Show that the metric induced in the Misner space by both (1) and (2) in (4.2) is

$$ds^2 = \frac{dT^2}{T} - 4T dz^2 . \tag{4.3}$$

This metric does not yet completely specify the Misner space. In order to complete the description of the Misner universe, one has to introduce a nontrivial topology. This is achieved by identifying points that satisfy two conditions: *(A)* they are associated to the same T coordinate and *(B)* they are related *via* a boost with velocity $v = \tanh(\psi_0)$ in the Minkowski space.

(d) *(2 points)* Show that the identification of points above leads to the identification of z-coordinates $z' \sim z + \psi_0$.
Hint: The following identities are useful: $\cosh(a)\cosh(b) \pm \sinh(a)\sinh(b) = \cosh(a \pm b)$ and $\sinh(a)\cosh(b) \pm \cosh(a)\sinh(b) = \sinh(a \pm b)$.

It is convenient to introduce an additional change of variable, that replaces the coordinate z with the coordinate ψ defined as,

$$\psi = z - \frac{1}{2}\log(|T|) . \tag{4.4}$$

(e) *(2 points)* Show that the change of variable (4.4) leads to the following metric for the Misner Universe:

$$ds^2 = -4dT d\psi - 4T d\psi^2 . \tag{4.5}$$

For which ranges of parameters T and ψ is the metric (4.5) valid? Show that the identification induced by the boosts explained above corresponds to the identification $\psi \sim \psi + \psi_0$. The metric (4.5), together with this identification, defines the Misner space.

Causality in the Misner Universe. In the following, we will study how the Misner Universe may allow for closed timelike curves.

(f) *(2 points)* Compute the light-cones in the (T, ψ)–coordinates and sketch them.

(g) *(2 points)* Show that a timelike geodesic that is in the Minkowski space described by the equation $x = x_0$, for some constant x_0, has the following functional form in the (T, ψ)-coordinates:

$$T(\psi) = x_0 e^{-\psi} - e^{-2\psi} . \tag{4.6}$$

(h) *(2 points + 2 bonus points)* Show that for an appropriate range of values of x_0 there can be geodesics that, starting at a point (T, ψ), they can end at an equivalent spacetime point (T', ψ'), such that $T' = T$ and $\psi' \neq \psi$. Based on what was said above, argue that such geodesics can be used to construct *closed timelike curves* (CTCs). Moreover, argue that the topology of the Misner space $\mathcal{M}$ is homeomorphic to that of $\mathbb{R} \times S^1$.

Hint: Solve Eq. (4.6) for $\psi = \psi(T, x_0)$ and then investigate for which values of T and x_0 the conditions for the existence of CTCs are satisfied. Write down explicitly these conditions.