DEPARTMENT OF PHYSICS AND ASTRONOMY, FACULTY OF SCIENCE, UU.
MADE AVAILABLE IN ELECTRONIC FORM BY THE $\mathcal{T}\!\mathcal{B}\!\mathcal{C}$ OF A–ESKWADRAAT
IN 2007/2008, THE COURSE NS-TP428M WAS GIVEN BY RENATE LOLL.

General Relativity (NS-TP428m)
February 1, 2008

You must obtain at least 10 of the 20 points to pass the exam. The use of auxiliary materials such as books, notes, calculators, laptops etc. is not permitted. Please hand in all sheets you used for calculations.

Question 1: Schwarzschild spacetime *(9 points)*

a) Starting from the spherically symmetric Schwarzschild solution to the vacuum Einstein equations in terms of standard Schwarzschild coordinates (t, r, θ, ϕ), define a new radial coordinate $r^*(r)$ by $dr^*/dr = 1/(1 - \frac{2GM}{r})$. Next, define a new time coordinate $u := t - r^*$, and compute the line element ds^2 of the Schwarzschild metric in terms of the coordinates (u, r, θ, ϕ). *(1 point)*

b) For a spacecraft falling freely toward the Schwarzschild singularity on a radial trajectory, compute its four-velocity V^μ as a function of r. Show that V_u, is constant along the trajectory. (The index u refers to the time direction u.) *(4 points)*

c) The spacecraft emits a monochromatic, outwardly radial light signal, whose wave length is measured as λ_{r_0} by a nearby stationary observer at fixed radius r_0. What is the wave length λ_∞ of the same signal when it is received by a distant stationary observer at very large, fixed radius $(r \to \infty)$? Determine first what is conserved along the photon's trajectory and then argue in terms of the photon energies seen by the observers. Are there any restrictions on r_0? *(4 points)*

Question 2: A two-dimensional spacetime *(11 points)*

Consider a two-dimensional spacetime metric with line element

$$ds^2 = -du^2 + f(u)^2 \, \mathrm{d}\phi^2, \tag{1}$$

with a real function $f(u)$ which is non-vanishing for all $u, -\infty < u < \infty$, and has a non-trivial dependence on u.

a) Compute the Riemann curvature tensor of the metric (1). How many components are algebraically independent? *(3 points)*

b) There are no Einstein equations in two dimensions, but with the ansatz there exist solutions $f_\Lambda(u)$ of the equations

$$R_{\mu\nu} + g_{\mu\nu}\Lambda = 0. \tag{2}$$

where $\Lambda \neq 0$ is a (cosmological) constant. Compute an explicit solution $f_\Lambda(u)$ for which the metric is well-defined for all times. *(3 points)*

c) Work out the Killing equations for the solution $f_\Lambda(u)$ found in b) and find the Killing vector field(s) K^μ of the corresponding spacetime. Verify Killing vector property by explicit calculation. (Hint: variable separation might come in handy.) *(5 points)*