Final exam for Cosmology (NS-TP430M)

From 10am-1pm on Wed July 3, 2013, in BBG 065.

In total: 40 points = 40% of the final grade + 5 bonus pts. It is a closed books, 3 hours exam.

Good luck!

1. Theoretical questions. *(8 points)*

(a) *(2 points) Big Bang Nucleosynthesis.*

Name the main isotopes produced at primordial nucleosynthesis (BBN). For which isotopes observations show agreement and for which disagreement with observations?

(b) *(2 points) Inflation.*

Inflation requires a "flat potential." Explain what a flat potential means, and which observation or observations demand it.

(c) *(2 points) QCD transition.*

What symmetry gets broken by the QCD chiral condensate in the early Universe. Explain!

(d) *(2 points) Cosmological perturbations.* The Planck satellite data constrain the spectrum of scalar cosmological perturbations to be nearly scale invariant (with a more than five standard deviations evidence for a deviation from scale invariance). Explain what kind of inflationary model would produce a scale invariant spectrum (to a very high precision). Is there any theoretical obstacle that forbids the spectrum to be exactly scale invariant?

2. An empty Universe with cosmological constant. *(6 points)*

Solve the Friedmann equation for an empty homogeneous expanding universe with a cosmological term Λ, for which the Friedmann equation is given by

$$H^2 \equiv \frac{\Lambda}{3} - \frac{c^2\kappa}{a^2}\,, \tag{1}$$

where κ denotes the curvature of the spatial section of space-time.

Discuss the solutions in both cases: when $\kappa = 1/R_{\rm curv}^2 > 0$ and when $\kappa = -1/R_{\rm curv}^2 < 0$, where $R_{\rm curv}$ denotes the (comoving) radius of curvature of the Universe. Show in particular that,

$$\Omega_\Lambda(t) - 1 = \frac{c^2\kappa}{(Ha)^2} = \frac{1}{\sinh^2\left(\sqrt{\Lambda/3}\,(t-t_0)\right)} \to 0\,, \qquad \text{when } a\to\infty \qquad (\kappa > 0) \tag{2}$$

and

$$\Omega_\Lambda(t) - 1 = -\frac{1}{\cosh^2\left(\sqrt{\Lambda/3}\,(t-t_0)\right)} \to 0\,, \qquad \text{when } a\to\infty \qquad (\kappa < 0)\,, \tag{3}$$

such that in the limit $a \to \infty$ the Universe becomes spatially flat. Discuss how this result can be used to solve the flatness problem of the Universe.

Hint: The following integrals may be useful:

$$\int \frac{dx}{\sqrt{x^2-1}} = \mathrm{Arcosh}(x)\,, \qquad \int \frac{dx}{\sqrt{x^2+1}} = \mathrm{Arsinh}(x)\,. \tag{4}$$

3. Relativistic degrees of freedom. *(4 points)*

Calculate the effective number of relativistic degrees of freedom g_* in the Minimal Supersymmetric Standard Model (MSSM) of elementary particles. Assume that all particles are relativistic (which is the case when $k_B T \gg E_{\rm ew} \simeq 100$ GeV), in thermal equilibrium at the same temperature.

Hint: Recall that in a supersymmetric theory the degrees of freedom are doubled, such that to each fermionic (bosonic) degree of freedom one adds a bosonic (fermionic) degree of freedom. In addition, the MSSM has two complex Higgs doublets.

4. The entropy of the Universe. *(4 points)*

Prove that the (thermodynamic) entropy of the Universe is conserved.

Hint: You may assume the expansion of the Universe to be adiabatic.

5. Slow roll inflation. *(10 points + 3 bonus points)*

Consider the slow roll regime of a single scalar field inflationary model whose action, Lagrangian and potential are,

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi\,, \qquad \mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu} - V(\phi)\,, \qquad V = V_0 e^{-\lambda\phi/M_P}\,. \tag{5}$$

In this problem work in units $\hbar = 1 = c$, in which the reduced Planck mass equals $M_{\rm P}^2 = 1/(8\pi G_N)$.

(a) *(3 points)* Starting with the equation of motion for the inflaton and the Friedmann equation written in the slow roll regime show that the inflaton (homogeneous background field) evolves as,

$$\phi(t) = \frac{2M_P}{\lambda}\ln\left(\frac{\lambda^2}{2M_P}\sqrt{(V_0/3)}\,t\right)\,. \tag{6}$$

(b) *(2 points)* Show next that

$$a = a_0\,(t/t_0)^{\frac{2}{\lambda^2}}\,, \qquad (a_0 = a(t_0))\,. \tag{7}$$

(c) *(1 point)* Show that the number of e-foldings

$$N(t) = -\frac{2}{\lambda^2}\ln(t/t_0) = -\ln(a/a_0)\,, \tag{8}$$

where t_0 denotes the end of inflation (at which $N = 0$).

(d) *(4 points)* Find an expression for the scalar spectral index (n_s) and for the tensor (graviton) spectral index (n_g) for this model. Recall their definitions,

$$\begin{aligned}\mathcal{P}_{w_\psi}(k) &= \mathcal{P}_{w_\psi *}\left(\frac{k}{k_*}\right)^{n_s-1}, \quad \mathcal{P}_{w_\psi *} = \frac{H_*^2}{8\pi^2\epsilon_* M_{\rm P}^2}, \quad n_s - 1 = -6\epsilon + 2\eta \\ \mathcal{P}_{\rm graviton}(k) &= \mathcal{P}_{\rm graviton*}\left(\frac{k}{k_*}\right)^{n_g}, \quad \mathcal{P}_{\rm graviton*} = \frac{2H_*^2}{\pi^2 M_{\rm P}^2}, \quad n_g = -2\epsilon\,,\end{aligned} \tag{9}$$

where a star ($*$) refers to the first Hubble crossing of a mode $k = k_*$ during inflation ($k_* = (Ha)_* \equiv H(t_*)a(t_*)$). Work at the leading order in slow roll approximation. Recall also the definitions

$$\epsilon = -\frac{\dot H}{H^2}\,, \qquad \eta = -\frac{\ddot\phi}{H\dot\phi} + \epsilon\,. \tag{10}$$

(e) *(3 bonus points)* This problem can be solved without making a slow roll approximation, *i.e.* the equations of motion for $\phi(t)$ and $H(t)$ can be simultaneously and exactly solved. Start with an Ansatz, $\phi(t) = (2M_{\rm P}/\lambda)\ln(\beta t)$ and find an equation for β. A useful quantity to introduce is $y = (\lambda^2 V_0)/(2M_{\rm P}^2\beta^2)$. Show that this quantity obeys the following quadratic equation,

$$y^2 + 2\left(1 - \frac{3}{\lambda^2}\right)y + \left(1 - \frac{6}{\lambda^2}\right) = 0\,. \tag{11}$$

There are two solutions, $y_\pm$ to this quadratic equation, of which only one – y_+ – is physical. Discuss the limit for the resulting β_+ and in which one obtains the slow roll solution from the first part of the problem. What is the condition on λ, which guarantees that one gets slow inflation? In addition, show that the solution of Eq. (11) implies $\epsilon = \lambda^2/2$, $\dot\epsilon = 0$, such that the slow roll approximation reproduces in this particular case an exact result for the scale factor (7).

6. Quantization of a massless scalar field in matter era. *(8 points + 2 bonus points)*

Consider a massless minimally coupled real scalar field with the action,

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\Big)\,, \tag{12}$$

where $g = \det[g_{\mu\nu}]$ is the determinant of the metric. In a FLRW cosmology the metric tensor, when expressed in conformal coordinates, is of the form,

$$g_{\mu\nu} = a^2\eta_{\mu\nu}\,, \qquad \eta_{\mu\nu} = {\rm diag}(1,-1,-1,-1)\,, \tag{13}$$

where $a = a(\eta)$ denotes the scale factor, and η is conformal time. In matter era the scale factor is of the form,

$$a(\eta) = a_0(\eta/\eta_0)^2\,, \qquad a_0 = a(\eta_0)\,. \tag{14}$$

(a) *(3 points)* By varying the action (12) in the space-time (13), show that the equation of motion for the scalar field reads,

$$\partial_\eta^2\phi + 2\frac{a'}{a}\partial_\eta\phi - \nabla^2\phi = 0\,, \qquad a' = \frac{da}{d\eta}\,, \quad \partial_\eta = \frac{\partial}{\partial\eta}\,. \tag{15}$$

The scalar field ϕ can be promoted to an operator, $\phi \to \hat{\phi}$, by the canonical quantization,

$$[\hat{\phi}(\vec{x},\eta), \hat{\pi}_\phi(\vec{x}\,',\eta)] = \imath\hbar\delta^3(\vec{x}-\vec{x}\,') \tag{16}$$

where $\hat{\pi}_\phi = a^2 d\hat{\phi}/d\eta$ denotes the canonical momentum of $\hat{\phi}$. This can be achieved by the following decomposition,

$$\hat{\phi}(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3} \mathrm{e}^{\imath\vec{k}\cdot\vec{x}} \Big[\varphi_k(\eta)\hat{a}_{\vec{k}} + \varphi_k^*(\eta)\hat{a}^\dagger_{-\vec{k}}\Big] , \tag{17}$$

where $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ denote the annihilation and creation operators, and φ_k and φ_k^* are the mode functions, which due to the homogeneity of the background space (one assumes a spatial translation invariance of the state), depend only on the modulus $k = \|\vec{k}\,\|$. The annihilation and creation operators satisfy the commutation relations,

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = \hbar(2\pi)^3\delta^3(\vec{k}-\vec{k}') , \qquad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 , \qquad [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = 0 . \tag{18}$$

In particular, the annihilation operator $\hat{a}_{\vec{k}}$ annihilates the vacuum, $\hat{a}_{\vec{k}}|\Omega\rangle = 0$, while the creation operator $\hat{a}^\dagger_{\vec{k}}$ creates one particle excitation of momentum $\vec{k}$ out of the vacuum, $\hat{a}^\dagger_{\vec{k}}|\Omega\rangle = |1_{\vec{k}}\rangle$. Show that the mode functions φ_k and $\varphi_k^\dagger$ obey the following Wronskian normalization condition,

$$\mathrm{W}[\varphi_\mathrm{k}, \varphi_\mathrm{k}^*] \equiv \varphi_k \frac{d\varphi_k^*}{d\eta} - \frac{d\varphi_k}{d\eta}\varphi_k^* = \frac{\imath}{a^2} . \tag{19}$$

(b) (*1 point*) Show that the mode functions φ_k in a FLRW space-time (13) obey the following equation of motion,

$$\left(\frac{d^2}{d\eta^2} + k^2 - \frac{a''}{a}\right)(a\varphi_k) = 0 , \quad (k = \|\vec{k}\|) . \tag{20}$$

(c) (*2 points*) Show that during matter era in which $a(\eta)$ is given in (14), Eq. (20) reduces to

$$\Big(\varphi_k'' + k^2 - \frac{2}{\eta^2}\Big)(a\varphi_k) = 0 , \tag{21}$$

and show further that the fundamental positive and negative frequency solutions are (the Bunch-Davies vacuum) are

$$\varphi_k = \frac{1}{a\sqrt{2k}}\Big(1 - \frac{\imath}{k\eta}\Big)\mathrm{e}^{-\imath k\eta} , \qquad \varphi_k^* = \frac{1}{a\sqrt{2k}}\Big(1 + \frac{\imath}{k\eta}\Big)\mathrm{e}^{\imath k\eta} . \tag{22}$$

These mode functions are identical in form to the solutions in de Sitter space. Show that the mode normalization, $1/(a\sqrt{2k})$, follows from the Wronskian (19).

(d) (*2 points*) The power spectrum $\mathcal{P}_\varphi$ of the scalar field fluctuations can be defined by

$$\langle\Omega|\hat{\phi}(\vec{x},\eta)\hat{\phi}(\vec{x}\,',\eta)|\Omega\rangle = \hbar\int\frac{d^3k}{(2\pi)^3}|\varphi_k|^2\mathrm{e}^{\imath\vec{k}\cdot\vec{r}} \equiv \int\frac{dk}{k}\mathcal{P}_\varphi(k,\eta)\frac{\sin(kr)}{kr} , \quad (\vec{r} = \vec{x}-\vec{x}\,' ,\ r = \|\vec{r}\,\|) . \tag{23}$$

Calculate the power spectrum in matter era and show that it can be written as

$$\mathcal{P}_\varphi(k,\eta) \;=\; \frac{\hbar k^2}{4\pi^2 a^2}\left\{1 + \frac{1}{(k\eta)^2}\right\} = \frac{\hbar H^2}{16\pi^2}\left\{1 + \left(\frac{2k}{aH}\right)^2\right\} . \tag{24}$$

Is this spectrum scale invariant on super-Hubble scales? Justify your answer.

(e) (*2 bonus points*) Based on your answer to question **(d)**, what do you think: do we still need inflation?