OBSERVATIONAL AND THEORETICAL COSMOLOGY

FINAL EXAM, 9am-12am on 29.06.2015 (in GAMMA of Educatorium)

You have 3 hours. You may have an A4 sheet of paper with formulae. No books are allowed.

45 points=45% of the grade in total. Good luck!

1 Theoretical Questions. *(12 points)*

Please answer concisely (in one or two sentences) the following questions:

(A) *(3 points)* Why is the sky dark at night? Give at least two reasons!

(B) *(3 points)* What are the principal shortcomings of the Newtonian derivation of the Friedmann equations?

(C) *(3 points)* Name all of the fermionic species of the standard model and write down (in parentheses) the number of relativistic degrees of freedom for each species.

(D) *(3 points)* You know that energy is not conserved in an expanding universe. What quantity is conserved in an expanding (homogeneous and isotropic) universe?

2 The line element on a 2-dimensional hyperboloid *(11 points)*

(A) *(4 points)* There is no 3-dimensional flat Euclidean embedding space for the 2-dimensional hyperboloid H^2. Instead, the embedding space is of a Lorentzian type,

$$dL_{L3}^2 = dY_1^2 + dY_2^2 - dY_3^2 \,, \tag{2.1}$$

the constraint is of a hyperboloidal type,

$$Y_1^2 + Y_2^2 - Y_3^2 = -R^2 \,, \tag{2.2}$$

where R is the throat radius of the hyperboloid. By explicitly finding suitable coordinate transformations, $Y_i = Y_i(\chi, \phi)$, $(i = 1, 2, 3)$ show that the line element on H^2 can be written as,

$$d\ell_{H2}^2 = d\chi^2 + R^2 \left[\sinh^2(\chi/R) d\phi^2\right] \,, \qquad (0 \leq \chi < \infty, 0 \leq \phi < 2\pi) \,. \tag{2.3}$$

Write down the metric tensor, g_{ij} that corresponds to the line element (2.3).

(B) *(3 points)* Show that, by the suitable coordinate transformation, the line element (2.3) can be written as,

$$d\ell^2 = \frac{dr^2}{1 - \kappa r^2} + r^2 d\phi^2 \,, \qquad (0 \leq r < \infty, 0 \leq \phi < 2\pi) \,. \tag{2.4}$$

Express κ in terms of R and explain its physical meaning. The form of the metric (2.4) is useful in that it is convenient to write down the metrics with zero and positive spatial curvature. How would you write down these two metrics in the (r, ϕ)-coordinates?

(C) *(4 points)* A negatively curved expanding (three dimensional) universe ($\kappa < 0$) has the metric of the form,

$$ds^2 = -c^2dt^2 + a(t)^2\left[d\chi^2 + R^2\sinh^2(\chi/R)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right],$$
$$(0 < t, 0 \le \chi < \infty, 0 \le \theta < \pi, 0 \le \phi < 2\pi). \qquad (2.5)$$

Solve the corresponding Friedmann equation for a highly relativistic fluid, for which the equation of state parameter, $w = 1/3$.

Hint: Show first that the energy density scales as, $\rho = \rho_0/a^4$, and then solve the Friedmann equation, $H^2 = [8\pi G_N/(3c^2)]\rho - \kappa c^2/a^2$. You may find it useful to work in conformal time η, defined by $d\eta = dt/a(t)$. Also, you may find useful the integrals listed at the end of problem 3 below.

3 Luminosity distance *(11 points + 4 bonus points)*

Angular diameter distance can be expressed in terms of the proper distance as,

$$d_L = (1+z)s_\kappa(d_{\text{prop}}(z)), \qquad (3.1)$$

where $d_{\text{prop}}(z) = c\int_{t(z)}^{t_0} dt/a(t) = R_{H0}\int_0^z d\bar{z}/[H(\bar{z})/H_0]$ and $s_\kappa(x) = x$ for $\kappa = 0$, $s_\kappa(x) = \sin(\sqrt{\kappa}x)/\sqrt{\kappa}$ for $\kappa > 0$ and $s_\kappa(x) = \sinh(\sqrt{-\kappa}x)/\sqrt{-\kappa}$ for $\kappa < 0$.

(A) *(6 points)* Calculate d_L as a function of z in a Universe which consists of a cosmological constant and is negatively spatially curved. Assume that the current content is $\Omega_{\Lambda,0}$ and $1 > \Omega_{\kappa,0} = 1 - \Omega_{\Lambda,0} > 0$. *Hint:* Show that,

$$d_{\text{prop}}(z) = \frac{c}{H_0\sqrt{\Omega_{\kappa,0}}}\ln\left(\frac{\sqrt{\Omega_{\kappa,0}}(1+z) + \sqrt{\Omega_{\kappa,0}(1+z)^2 + \Omega_{\Lambda,0}}}{1+\sqrt{\Omega_{\kappa,0}}}\right). \qquad (3.2)$$

What do you get for $d_{\text{prop}}(z)$ in the limit when $\Omega_{\kappa,0} \to 0$ and what in the limit when $\Omega_{\Lambda,0} \to 0$?

(B) *(5 points)* Sketch $d_L(z)$ and $d_{\text{prop}}(z)$ as a function of z. Does $d_L(z)$ ($d_{\text{prop}}(z)$) grow to infinity as $z \to \infty$, or it remains finite?

(C) *(4 bonus points)* Assume that $\Omega_{\kappa,0} < 0$ and sketch $d_L(z)$ in this case! What is the limit of $d_L(z)$ ($d_{\text{prop}}(z)$) when z grows large in this case (note that z cannot grow to infinity)? Can you explain the origin of the difference between the cases (B) and (C)! Pay attention to the fact that there is a maximum redshift allowed.

Hint: You may find useful the following integrals:

$$\int\frac{dx}{\sqrt{1+x^2}} = \ln[x+\sqrt{1+x^2}] = \text{Arcsinh}[x], \qquad \int\frac{dx}{\sqrt{1-x^2}} = \text{Arcsin}[x]$$

4 Perturbations in Einstein de Sitter space *(11 points + 3 bonus points)*

Consider an Einstein de Sitter universe, namely a universe with only non-relativistic matter, $\Omega_{m,0} = 1$.

(A) *(6 points)* Consider the continuity and Euler equations for a pressureless fluid:

$$\partial_t \rho + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0 \tag{4.1}$$

$$\partial_t \vec{V} + \left(\vec{V} \cdot \nabla\right) \vec{V} + \vec{\nabla}\phi = 0 . \tag{4.2}$$

Assume adiabatic perturbations, namely $\delta S = 0$ and $\vec{V} = \vec{\nabla} u$ with $\vec{\nabla} \cdot \vec{V} = \vec{\nabla} \cdot \vec{\nabla} u \equiv \theta$ and the background

$$\rho = \bar{\rho}(t)\left[1 + \delta(\vec{x}, t)\right] , \quad \vec{V} = H\vec{x} + \vec{v} . \tag{4.3}$$

Expand the continuity equation to zeroth order in perturbations and find the time dependence of $\bar{\rho}(t)$. Using the Friedmann equation, derive the time dependence of $H(t)$ and that of $a(t)$. Expand both the continuity and the Euler equations to linear order in perturbations.

(B) *(2 points)* Define comoving coordinates according to $\vec{x} = a(t)\vec{q}$. Rewrite the linear equations in comoving coordinates by deriving the time and space derivatives in comoving coordinates $\partial_t|_q$ and $\partial_q|_t$

(C) *(3 points + 3 bonus points)* Consider the linearized continuity, Euler and Poisson equations in comoving coordinates

$$\partial_t \delta + \frac{1}{a}\vec{\nabla}\vec{v} = 0 , \tag{4.4}$$

$$\partial_t \vec{v} + H\vec{v} + \frac{1}{a}\vec{\nabla}\phi = 0 , \tag{4.5}$$

$$\nabla^2 \phi = 4\pi G_N a^2 \bar{\rho}\delta . \tag{4.6}$$

Derive a differential equation for θ alone (without any δ or ϕ). Solve this equation to obtain the time dependence of the velocity divergence θ. From the continuity equation show that the growing solution for θ leads to the growing solution $\delta \propto a$. Using the Poisson equation, find the time dependence of ϕ.

Hint: to solve the differential equation for θ you can use an ansatz $\theta \propto t^\alpha$ and try to determine the two solutions for α.

Bonus points indicate that the question is a bit more difficult to solve. You do not need to solve bonus questions to get full credit. If you do, they will count towards full credit in that particular problem.