INSTITUTE FOR THEORETICAL PHYSICS, FACULTY OF PHYSICS AND ASTRONOMY, UU.
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String Theory (NS-TP526M)
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Question 1. $L_0 - \tilde{L}_0$

We have seen that in conformal guage, the Hamiltonian for the closed string is related to the generators of the Virasoro algebra by the identity $H = (L_0 + \tilde{L})_0)/2$. Therefore, $L_0 + \tilde{L}_0$ generates time translations on the worldsheet along the time coordinate τ. Similarly, for the closed string, one can express the translations along the spatial coordinate σ in terms of $L_0 - \tilde{L}_0$. This follows from the identity

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma} = [-i(L_0 - \tilde{L}_0, X^\mu]. \tag{1}$$

a) Prove this relation, using the identities given in the formularium.

Question 2. String propagator in light cone guage

Consider the transverse components of the open string operator $X^i(\tau,\sigma)$ in light-cone guage. We wish to compute the propagator or two-point correlation function, associated with the transverse modes of an open string. Normal ordering on the oscillator modes is defined by putting all annihilation operators $\alpha^i_n, n > 0$ to the right of the creation operators. Let us extend the definition of normal ordering to the position and momentum operators by: $p^i x^j := x^j p^i$.

a) Show first, using the notation and identities in the formularium for the open string, that

$$[A^i(\tau,\sigma), A^{j\dagger}(\tau',\sigma')] = \delta^{ij} G(\tau,\tau',\sigma,\sigma'), \tag{2}$$

where, with the usual worldsheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$,

$$G(\tau,\tau',\sigma,\sigma') = -\tfrac{1}{4}\log\left[(e^{i\sigma'^+} - e^{i\sigma^+})(e^{i\sigma'^-} - e^{i\sigma^-})\right] + R(\tau,\tau',\sigma,\sigma'), \tag{3}$$

and $R(\tau,\tau',\sigma,\sigma')$ denotes terms which are regular in the limit $\tau \to \tau', \sigma \to \sigma'$.
Find the expression for G and therefore of R.
Hint: you may use the identity

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x). \tag{4}$$

Consequently, show that

$$X^i(\tau,\sigma)X^j(\tau',\sigma') =: X^i(\tau,\sigma)X^j(\tau',\sigma') : +\delta^{ij}G(\tau,\tau',\sigma,\sigma'), \tag{5}$$

where G is again given by (3), but with a different regular term R'. Give the expression of R' in terms of R.

Question 3. The graviton

In the closed string spectrum in light-cone guage, we have found a physical state

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle, \tag{6}$$

where i and j run over the transverse directions, $i,j = 1,\ldots,D-2$.

a) Show that the traceless-symmetric part of this state corresponds to the physical components of a massless graviton in D dimensions. Do this by matching the number of physical degrees of freedom.

4. Formularium

4.1. Closed string

The oscillator expansion for the closed string in conformal guage reads (in units where the string tension is taken $T = 1/4\pi$),

$$X^{\mu}(\tau,\sigma) = X_R^{\mu}(\tau-\sigma) + X_L^{\mu}(\tau+\sigma), \tag{7}$$

where

$$\begin{aligned} X_R^{\mu}(\tau-\sigma) &= \tfrac{1}{2}x^{\mu} + p^{\mu}(\tau-\sigma) + i\sum_{n\neq 0}\frac{\alpha_n^{\mu}}{n}e^{-in(\tau-\sigma)}, \\ X_L^{\mu}(\tau+\sigma) &= \tfrac{1}{2}x^{\mu} + p^{\mu}(\tau+\sigma) + i\sum_{n\neq 0}\frac{\tilde{\alpha}_n^{\mu}}{n}e^{-in(\tau+\sigma)}. \end{aligned} \tag{8}$$

The Virasoro generators are normal ordered as

$$L_m = \tfrac{1}{2}\left(\sum_{n=-\infty}^{+\infty} :\alpha_{m-n}^{\mu}\alpha_{n,\mu}: - a\delta_{m,0}\right); \qquad \tilde{L}_m = \tfrac{1}{2}\left(\sum_{n=-\infty}^{+\infty} :\tilde{\alpha}_{m-n}^{\mu}\tilde{\alpha}_{n,\mu}: - a\delta_{m,0}\right), \tag{9}$$

with $\alpha_0^{\mu} = \tilde{\alpha}_0^{\mu} = p^{\mu}$. The non-vanishing commutation relations in conformal guage are (we set $\hbar = 1$)

$$[x^{\mu}, p^{\nu}] = i\eta^{\mu\nu}, \qquad [\alpha_m^{\mu}, \alpha_n^{\nu}] = [\tilde{\alpha}_m^{\mu}, \tilde{\alpha}_n^{\nu}] = m\delta_{m+n,0}\eta^{\mu\nu}. \tag{10}$$

4.2. Open string in light-cone

We can write the oscillator expansion for the open string in light-cone guage as

$$X^i(\tau,\sigma) = x^i + p^i\tau + A^i(\tau,\sigma) + A^{i\dagger}(\tau,\sigma), \tag{11}$$

with

$$A^i(\tau,\sigma) = i\sum_{n=1}^{\infty}\frac{\alpha_n^i}{n}e^{-in\tau}\cos(n\sigma), \qquad A^{i\dagger}(\tau,\sigma) = -i\sum_{n=1}^{\infty}\frac{\alpha_{-n}^i}{n}e^{in\tau}\cos(n\sigma), \tag{12}$$

with the usual commutation relations

$$[x^i, p^j] = i\delta^{ij}, \qquad [\alpha_m^i, \alpha_n^j] = m\delta_{m+n,0}\delta^{ij} \tag{13}$$