INSTITUUT VOOR THEORETISCHE FYSICA
UNIVERSITEIT UTRECHT

MIDTERM EXAM String Theory

Tuesday, April 16, 2013.

1) Start every exercise on a separate sheet.

2) Write on each sheet: your name and initials, and your student number.

3) Please write legibly and clear.

4) The exam consists of **three** exercises. You are allowed to use lecture notes, books or any other handwritten or printed document but not computers or calculators, nor the internet.

1. Unoriented open strings

Consider the unoriented open string in D dimensions. Recall the definition: Consider the world-sheet parity transformation $T^\dagger X^\mu(\sigma,\tau)T = X^\mu(\pi-\sigma,\tau)$ where X^μ is the open string field,

$$X^\mu(\sigma,\tau) = x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos(n\sigma)\,.$$

Unoriented open strings satisfy $T^\dagger X^\mu(\sigma,\tau)T = X^\mu(\sigma,\tau)$.
a. Show that this implies $T^\dagger\alpha_n^\mu T = (-1)^n\alpha_n^\mu$. The spectrum of unoriented open-string is constructed *only* out of the oscillators α_n^μ that satisfy this relation. All other oscillators are thrown out of the spectrum. Which oscillator modes are left in the spectrum?

The rest of this problem is about the spectrum of the unoriented open string in the light-cone gauge.

b. The mass formula for the bosonic open string in the light-cone gauge reads

$$m^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i - a\right) \equiv \frac{1}{\alpha'}(N-a)\,.$$

Index i is summed over the transverse directions. Starting from this expression write down the mass formula for the unoriented open string.

c. Write down all the string states in the first three levels $N = 0$, $N = 1$ and $N = 2$. N is defined above. (Note that some of the levels may be empty).
What are their masses? (Note that it will depend on the normal ordering constant a.)
What is the dimensionality of the states at these levels (how many independent states are there in each level) .
d. Demanding that the little group is $SO(D-1)$ for massive states and $SO(D-2)$ for massless states find the value of a. In practice, this means that the total number of states at a given level should be either of $1, D-1, \cdots$ for massive states or $1, D-2, \cdots$ for massless states.
(If you cannot manage to obtain a leave it undetermined in the rest of the problem.)
e. The normal ordering constant a is also defined by the expression,

$$\sum_{n=-\infty}^{\infty} \alpha^i_{-n}\alpha^i_n = \sum_{n=-\infty}^{\infty} :\alpha^i_{-n}\alpha^i_n: -2a\,.$$

Starting from this and using the commutation relations of α^i_n and the unoriented string condition in (**a.**) above, evaluate a. In the calculation you will encounter a divergent sum of the form $\sum_{n=1}^{\infty}$. Define it by regularization to be $\sum_{n=1}^{\infty} \to -\frac{1}{12}$ as was done in class. Equating to the value of a you obtained in **d.** obtain the value of D.
f. Work out the spectrum at levels $N = 3$ and $N = 4$. How many states are there at each level? (If you could not determine D above leave it undetermined in your expressions, you will still get full credit.)
g. Show that the total number of states at the level $N = 4$ is the same as the number of states of a traceless symmetric tensor of $SO(D-1)$. Is this consistent with the mass formula you obtained in **b.** above?

2. Neumann and Dirichlet boundary conditions

The Polyakov action in the conformal gauge reads

$$S_P = \frac{T}{2}\int_0^{\bar{\sigma}} d\sigma \int_{\tau_0}^{\tau_f} d\tau\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\,.$$

a. Consider the variation of this action in fields X^μ, $\mu = 0, 1, \cdots 25$. At initial and final world-sheet times you have to impose $\delta X^\mu(\sigma,\tau_0) = \delta X^\mu(\sigma,\tau_f)$ as in the notes. Show that one possible solution to this variational problem is given by the field equations $(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0$ *together with* the boundary conditions:

$$\begin{aligned}
\partial_\sigma X^\mu(0,\tau) &= \partial_\sigma X^\mu(\bar{\sigma},\tau) = 0, \quad \mu = 0, 1, \cdots p\,, \\
\delta X^\mu(0,\tau) &= \delta X^\mu(\bar{\sigma},\tau) = 0, \quad \mu = p+1, p+2, \cdots 25\,.
\end{aligned}$$

b. This solution does not respect the original Lorentz symmetry of the Polyakov action $SO(1,25)$. Guess the space-time symmetry that is respected by this solution. (Hint:

Just consider the symmetry of the end-points of the solution under rotations among the indices μ.)

c. Set $\bar{\sigma} = \pi$ and derive the oscillator expansion for $X^\mu(\sigma, \tau)$ that satisfies the field equation and the boundary conditions above.

3. A classical string

Consider the following classical string configuration in 2+1 dimensions:

$$\begin{aligned} X^0 &= \kappa\tau\,, \\ X^1 &= c_1\sigma + A_1\cos(n_1\tau)\cos(n_2\sigma)\,, \\ X^2 &= c_2\sigma + A_2\cos(n_3\tau)\cancel{\cos}(n_4\sigma)\,, \end{aligned}$$

sin

where $\kappa, A_1, A_2, c_1, c_2$ and n_i are real numbers.

a. Under what conditions this configuration describes a *solution* to the Polyakov action

$$S_P = \frac{T}{2}\int_0^{2\pi} d\sigma \int_{\tau_0}^{\tau_f} d\tau\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\,,$$

that is *periodic in* σ.

b. Under what conditions this configuration satisfy the Virasoro constraints $T_{++} = T_{--} = 0$?

c. Set the parameters such that both **a** and **b** hold. What is the energy of the string? What are the momenta P^1 and P^2? What is the angular momentum J_{12}?

d. What is the motion of the string in τ? Depict the shape of the string at times $\tau = 0, \pi/4, \pi/2$ and π.