INSTITUUT VOOR THEORETISCHE FYSICA
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MIDTERM EXAM String Theory

Tuesday, April 15, 2014.

1) Start every exercise on a separate sheet.
2) Write on each sheet: your name and initials, and your student number.
3) Please write legibly and clear.
4) The exam consists of **two** exercises. You are allowed to use lecture notes, books or any other handwritten or printed document but not computers or calculators, nor the internet.

1. Open strings with mixed boundary conditions (70 points)

The Polyakov action in the conformal gauge reads

$$S_P = \frac{T}{2}\int_0^{\sigma_f} d\sigma \int_{\tau_i}^{\tau_f} d\tau\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\,.$$

a. (5 points) Consider the variation of this action in fields X^μ, $\mu = 0, 1, \cdots D-1$. At initial and final world-sheet times you have to impose $\delta X^\mu(\sigma,\tau_0) = \delta X^\mu(\sigma,\tau_f)$ as in the notes. Show that *the most general solution* to this variational problem is given by the field equations $(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0$ *together with* the mixed Neumann and Dirichlet boundary conditions that can be chosen differently at the two ends of the string $\sigma = 0$ and $\sigma = \sigma_f$:

$$\begin{aligned}
\partial_\sigma X^\mu(0,\tau) &= 0 \quad \mu = 0, 1, \cdots p_1\,,\\
\delta X^\mu(0,\tau) &= 0, \quad \mu = p_1+1, p_1+2, \cdots D-1\,,\\
\partial_\sigma X^\mu(\sigma_f,\tau) &= 0 \quad \mu = 0, 1, \cdots p_2\,,\\
\delta X^\mu(\sigma_f,\tau) &= 0, \quad \mu = p_2+1, p_2+2, \cdots D-1\,,
\end{aligned}$$

With no loss of generality one can take $D-1 \geq p_1 \geq p_2$. Assume this below.

b. (5 points) The second and last equations above just means that the end point of the string in the corresponding directions are fixed in space. Take these locations as

$$\begin{aligned} X^\mu(0,\tau) &= x_i^\mu, \quad \mu = p_1+1, p_1+2, \cdots D-1\,, \\ X^\mu(\sigma_f,\tau) &= x_f^\mu, \quad \mu = p_2+1, p_2+2, \cdots D-1\,, \end{aligned}$$

This solution does not respect the original Lorentz symmetry of the Polyakov action $SO(1,25)$. Guess the space-time symmetry that is respected by this solution. (Hint: Think about the symmetry of the end-points of the solution under rotations among the indices μ.)

c. (10 points) Set $\sigma_f = \pi/2$. *Note that this is different than the usual open string choice of* $\sigma_f = \pi$; *this choice will be more convenient here.* The most general solution $X^\mu(\sigma,\tau)$ that satisfies the field equation is $X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$, and the left/right sections can be expanded in Fourier components as usual:

$$\begin{aligned} X_R^\mu &= \frac{1}{2}x^\mu + b\,p_R^\mu(\tau-\sigma) + \frac{i}{2\sqrt{\pi T}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)}\,, \\ X_L^\mu &= \frac{1}{2}x^\mu + b\,p_L^\mu(\tau+\sigma) + \frac{i}{2\sqrt{\pi T}}\sum_{n\neq 0}\frac{\overline{\alpha}_n^\mu}{n}e^{-in(\tau+\sigma)}\,. \end{aligned}$$

where $b \in R$ is a constant to be determined later.

By imposing the boundary conditions above find x^μ, p_L^μ and p_R^μ for each μ. Find how $\overline{\alpha}^\mu$ is related to α^μ and the condition on the integers n (e.g. are they odd, even or general integers?) for each μ.

Hint: You will have to group the fields into three classes with $\mu = 0, \cdots p_2$, $\mu = p_2+1, \cdots p_1$ and $\mu = p_1+1, \cdots D-1$.

d. (5 points) What does the reality condition $X^\mu = X^{\mu\dagger}$ implies?

e. (5 points) Calculate the center-of-mass momentum using the formula

$$\hat{P}^\mu = T\int_0^{\sigma_f} d\sigma\,\partial_\tau X^\mu(\sigma,0)\,. \tag{1}$$

Determine the constant b by requiring $\hat{P}^\mu = p_L^\mu + p_R^\mu$.

f. (5 points) *Interpretation in terms of D-branes:* As noted in class, if an end-point of the string has Neumann boundary conditions in the directions $\mu = 0, 1, \cdots p$ and Dirichlet conditions in the rest, this end point is said to be attached to a Dp brane. How do you interpret the general string configuration above in terms of D-branes?

g. (30 points) Now quantize this string in the *light-cone gauge* by following the steps below:

g1. The light cone gauge is

$$X^{+} = \kappa\tau\,, \ \ \text{where} \qquad X^{\pm} \equiv \frac{1}{\sqrt{2}}\left(X^{0} \pm X^{r}\right)\,, \tag{2}$$

where r is one of the spatial directions. What would you choose for r?
Hint: r should be chosen such that the center-of-mass mode of the string can move in the rth direction.
g2. What is κ? (If you do not know how to fix this, leave it as a free parameter below, you'll still get full credit from the questions below.)
g3. What is the *constrained* Hilbert space of this string? What is the vacuum state? How does the most general excited state looks like?
g4. Work out the constraint equation to obtain α_n^{-} in terms of the transverse oscillators. You should allow for a normal-ordering constant a defined as

$$\sum_m \alpha^i_{-m}\alpha^i_m = \sum_m :\alpha^i_{-m}\alpha^i_m: -2a\,, \tag{3}$$

where i runs over the transverse directions i.e. it can be anything except 0 and r.
g5. Now work out the mass formula for this string using $m^2 = -p^\mu p_\mu$. That is, obtain a formula for m^2 in terms of the transverse oscillators and the constant a.
g6. Calculate the normal ordering constant a in terms of D, p_1 and p_2. You should treat the divergent sums by setting

$$\sum_{m=1}^{\infty} m \to -\frac{1}{12}\,, \tag{4}$$

as in the notes. (Do not derive this, just use it).
g7. What is the mass of the vacuum state?
g8. What is the next lowest mass state? What is its mass?
g9. Write down all possible string states at the next level, i.e. level 2.

h. (5 points) A string is said to be *un-oriented* if it satisfies the condition

$$X^\mu(\sigma_f - \sigma, \tau) = X^\mu(\sigma, \tau)\,. \tag{5}$$

Now consider the case $p_1 = D-1$, $p_2 = 0$ namely it has Neumann boundary conditions at one end and Dirichlet conditions at the other end. On top of this, impose the *un-oriented string condition* above. What are the non-trivial excitations α_n^μ of this string?

2. A classical closed string (30 points)

Consider the following classical closed string configuration in 4+1 Minkowski space-time:

$$\begin{aligned} X^0 &= \kappa\tau\,, \\ X^1 &= a\sin n\sigma\cos n\tau\,, \\ X^2 &= a\sin n\sigma\sin n\tau \\ X^3 &= b\sin m\sigma\cos m\tau \\ X^4 &= b\sin m\sigma\sin m\tau\,. \end{aligned}$$

Here n, m are integers.

a. (12 points) Derive the relationship between a, n, b, m and κ required to satisfy the *constraints*.

b. (12 points) Compute the energy of the string and the angular momenta $J \equiv J_{12}$ and $\tilde{J} \equiv J_{34}$ corresponding to rotation of the string in spatial planes 12 and 34 respectively.

c. (6 points) Show that the energy is related to the angular momenta as

$$E = \sqrt{\frac{2}{\alpha'}(nJ + m\tilde{J})} \quad \textit{where} \qquad \alpha' = \frac{1}{2\pi T}\,. \tag{6}$$