INSTITUUT VOOR THEORETISCHE FYSICA
UNIVERSITEIT UTRECHT

FINAL EXAM String Theory

Friday, July 1, 2014.

1) Start every exercise on a separate sheet.

2) Write on each sheet: your name and initials, and your student number.

3) Please write legibly and clear.

4) The exam consists of **three** exercises. You are allowed to use lecture notes, books or any other handwritten or printed document but neither computers nor the internet.

1. Superstring Hamiltonian and the Virasoro algebra

Consider the superstring action in the superconformal gauge and in the light-cone coordinates $\tau^{\pm} = \tau \pm \sigma$ and $\psi^T = (\psi_+, \psi_-)$ as

$$S = 2T \int d^2\tau \left[\partial_+ X \cdot \partial_- X + i\left(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-\right)\right] .$$

We can write the action as $S = \int d^2\tau \mathcal{L}$ with Lagrangian density $\mathcal{L}$. The Hamiltonian density then follows from the general formula $\mathcal{H} = p\dot{q} - \mathcal{L}$, where the momentum p conjugate to q is given by $p = \partial\mathcal{L}/\partial\dot{q}$. The Hamiltonian H is then given by the spatial integral of H.

a. Obtain the Hamiltonian density following the procedure above.

b. For the open superstring the Hamiltonian is given by $H = \int_0^\pi d\sigma \mathcal{H}$. The oscillator expansions read

$$X^\mu(\tau, \sigma) = x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau} \cos(n\sigma) ,$$

$$\psi_\pm^\mu(\tau, \sigma) = \frac{1}{2\sqrt{\pi T}} \sum_{r\in Z+\delta} b_r^\mu e^{-ir\tau^\pm}$$

where r runs over integers in the Ramond sector $\delta = 0$ and over half integers $\delta = 1/2$ in the Neveu-Schwarz sector. Find the expression for the Hamiltonian in terms of the oscillator expansions, defining $\alpha_0^\mu = \frac{1}{\sqrt{\pi T}} p^\mu$.

c. The energy-momentum tensor and the super current that follows from the superstring action can be written as (I changed an overall normalization with no effect on the physics)

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X^\mu - \frac{1}{4}\partial^\gamma X^\mu \partial_\gamma X^\mu \eta_{\alpha\beta} + \frac{i}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \frac{i}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu\,,$$

$$G_\alpha = \frac{1}{4}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu\,.$$

Here ρ_α are the 2D Gamma-matrices that satisfy the Clifford algebra $\{\rho_a, \rho_b\} = 2\eta_{\alpha\beta}$. You will not need explicit expressions for the ρ_α in what follows.

c. By using the equations of motion in the super-conformal gauge, show that both the energy-momentum tensor and the super-current is conserved. **d.** Show that the conservation equation in the light-cone variables lead to infinitely many of conserved charges.

e. Show that the super-current further obeys the "tracelessness" condition:

$$\rho^\alpha G_\alpha = 0\,.$$

2. Superstring spectrum

a. Consider the open superstring, quantized in the light-cone gauge, in which the only independent oscillator modes are transversal and denoted by α^i_n and b^i_r where $i = 1, 2\cdots 8$. In the quantum theory, one imposes the commutation relations $[\alpha^i_m, \alpha^j_n] = m\delta_{m+n,0}\delta^{ij}$, and anti-commutation relations $\{b^i_r, b^j_s\} = \delta^{ij}\delta_{r+s}$. The mass operator is given by

$$\alpha' M^2 = \sum_{m=1}^{\infty} \alpha^i_{-m}\alpha^i_m + \sum_{r\in Z+\delta} r b^i_{-r} b^i_r - a_\delta\,, \tag{1}$$

where the normal ordering constant $a_{\frac{1}{2}} = 1/2$ in the Neveu-Schwarz (NS) sector, and $a_0 = 0$ in the Ramond (R) sector.

a. Construct the states in the NS sector of the open superstring spectrum, at mass level $\alpha' M^2 = 3/2$. Answer the following questions:

- How many degrees of freedom are there in these states? What is the total number of degrees of freedom you found?
- In total they should form a representation of the little group of $SO(1,9)$. What is this little group in this particular case?
- Are the states you found correspond to space-time bosons or fermions?
- The GSO operator in the NS sector is defined by $G_{NS} = -(-1)^F$, where $F = \sum_{r>0} b^i_{-r} b^i_r$. Which states you found above survive the GSO projection $G_{NS} = +1$?

b. Construct the states in the R sector of the open superstring spectrum, at mass level $\alpha' M^2 = 2$. Consider both of the possibilities with the Ramond ground state left handed and right handed, i.e. 8_c (negative chirality) and 8_s (positive chirality) in the notation of the notes. Answer the same questions as above:

- How many degrees of freedom are there in these states? What is the total number of degrees of freedom you found?
- In total they should form a representation of the little group of $SO(1,9)$. What is this little group in this particular case?
- Are the states you found correspond to space-time bosons or fermions?
- The GSO operator in the R sector is defined by $G_R = \Gamma^9(-1)^F$, where $\Gamma^9 = \pm 1$ measures the chirality of spinors in eight dimensions where $F = \sum_{r>0} b^i_{-r} b^i_r$. Which states you found above survive the GSO projection $G_R = +1$? Which ones survive $G_R = -1$?

c. In the closed bosonic string we showed that the most interesting level was $N = \overline{N} = 1$ with the state $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle$ contain the graviton $h_{\mu\nu}$ the B-field $B_{\mu\nu}$ and the dilation ϕ. Now consider the closed superstring and answer the following questions:

- What are the corresponding states in the closed superstring? Write down the states corresponding to the graviton $h_{\mu\nu}$ the B-field $B_{\mu\nu}$ and the dilation ϕ.
- Which sector do they belong to? (Note that there are four sectors now corresponding to NS or R for the left-movers and right-movers).
- Does they survive the GSO projection $G_{NS} = 1$, $G_R = 1$?
- Does they survive the GSO projection $G_{NS} = 1$, $G_R = -1$?
- **[Bonus]**. Can you think of a projection that keeps $h_{\mu\nu}$ but gets rid off $B_{\mu\nu}$?

3. Spiky strings

Consider the classical bosonic string propagating according to

$$X^0 = t = \tau\,,$$
$$\vec{X} = \vec{X}(\tau^+) + \vec{X}(\tau^-)\,.$$

Here $\vec{X} = \{X^i\}$, $i = 1, \cdots D-1$ and

$$\vec{X}(\tau^-) = \frac{\sin(m\tau^-)}{2m}\mathbf{e}_1 + \frac{\cos(m\tau^-)}{2m}\mathbf{e}_2\,,$$

$$\vec{X}(\tau^+) = \frac{\sin(n\tau^+)}{2n}\mathbf{e}_1 + \frac{\cos(n\tau^+)}{2n}\mathbf{e}_2\,,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two unit orthogonal vectors and the ratio $\frac{n}{m}$ is an integer. Answer the following questions.

- Show that this configuration satisfies the Virasoro constraints.
- Show that there are points on the string where $\vec{X}' = 0$. Show that at these points $\dot{\vec{X}}^2 = 1$ i.e. these points move with the speed of light - these are called *spikes*.
- Let $m = 1$ and $n = k - 1$. Show that k is the number of spikes.