Instituut voor Theoretische Fysica
Universiteit Utrecht

FINAL EXAM String Theory

Tuesday, June 28, 2016.

1) Start every exercise on a separate sheet.
2) Write on each sheet: your name and initials, and your student number.
3) Please write legibly and clear.
4) The exam consists of **four** exercises. You are allowed to use lecture notes, books or any other handwritten or printed document but neither computers nor the internet.

1. Superstring Hamiltonian (20 points)

Consider the superstring action in the superconformal gauge and in the light-cone coordinates $\tau^{\pm} = \tau \pm \sigma$ and $\psi^T = (\psi_+, \psi_-)$ as

$$S = 2T \int d^2\tau \, [\partial_+ X \cdot \partial_- X + i\,(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-)] \, .$$

We can write the action as $S = \int d^2\tau \mathcal{L}$ with Lagrangian density $\mathcal{L}$. The Hamiltonian density then follows from the general formula $\mathcal{H} = p\dot{q} - \mathcal{L}$, where the momentum p conjugate to q is given by $p = \partial \mathcal{L}/\partial \dot{q}$. The Hamiltonian H is then given by the spatial integral of H.

a. [10 points] Obtain the Hamiltonian density following the procedure above.

b. [10 points] Show that the "supercurrent"

$$G_\alpha = \frac{1}{4} \rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu \, ,$$

is conserved. Here ρ^α are the 2D gamma matrices given in the lecture notes. (You can show this either in these coordinates or by passing to the light-cone coordinates, using the 2D gamma matrices ρ^α given in the lecture notes.)

2. Fermions and OPEs (25 points)

World-sheet fermions satisfy the anti-commutation relations

$$\{\Psi^\mu(\sigma,\tau),\Psi^\nu(\sigma',\tau)\} = 2\pi\eta^{\mu\nu}\delta(\sigma-\sigma')$$

where σ and τ are the coordinates on the cylinder.

a. **[5 points]** Consider now the coordinates on the complex plane $z=\exp(\tau - i\sigma)$ and $\bar{z}=\exp(\tau+i\sigma)$. Write down what this anti-commutator implies for the radially ordered OPE of the holomorphic part $\Psi^\mu(z)$, $R[\Psi^\mu(z)\Psi^\nu(z')]$ on the complex plane.

b. **[10 points]** Ignore the space-time index μ for below. Using your answer in **a.**, calculate the OPE $R[T(z)\Psi(z')]$ where T is the holomorphic part of the superstring energy-momentum tensor:

$$T(z) = \frac{1}{2}:\partial_z X\partial_z X: + \frac{i}{2}:\Psi(z)\partial_z\Psi(z):\,.$$

If you cannot find the answer then guess the form of this OPE.

c. **[10 points]** Consider a fermion $\Psi(\sigma,\tau)$ on the cylinder (σ,τ) where σ is the periodic coordinate with period 2π. Suppose that this fermion satisfies generalized periodicity conditions:

$$\Psi(\sigma+2\pi,\tau) = e^{2\pi i\nu}\Psi(\sigma,\tau)\,,$$

where $\nu\in[0,2\pi)$ is a real number. Derive the oscillator expansion for this field.

3. Superstring spectrum (25 points + 5 bonus)

The mass operator (in dimensionless units) for the open superstring is given by

$$\alpha' M^2 = \sum_{m=1}^{\infty}\alpha^i_{-m}\alpha^i_m + \sum_{r\in Z+\delta>0} r b^i_{-r}b^i_r - a_\delta\,, \tag{1}$$

where the normal ordering constant $a_{\delta=1/2}=1/2$ in the Neveu-Schwarz (NS) sector, and $a_{\delta=0}=0$ in the Ramond (R) sector.

a. **[10 points]** Construct the states in the NS sector of the open superstring spectrum, at mass level $\alpha' M^2=+3/2$ and $\alpha' M^2=+2$. Answer the following questions:

- List all possible states you can construct at these mass levels.
- How many degrees of freedom are there in these states? (You do not have to add up all the numbers you found, you can just leave them as a sum of products.)

$-G_{00}(r)dt^2 + G_{rr}(r)dr^2 + G_{ij}(r)dX^i dX^j$ with $i = 1, \cdots D-2$ and r is just another space dimension. The Nambu-Goto action on $G_{\mu\nu}$ is defined as usual:

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-det h_{\alpha\beta}} \tag{2}$$

with

$$h_{\alpha\beta} = \frac{\partial X^\mu}{\partial \tau^\alpha} \frac{\partial X^\nu}{\partial \tau^\beta} G_{\mu\nu} \,. \tag{3}$$

Here $X^\mu = (X^0, r, X^1, \cdots, X^{D-2})$ is the position vector in the whole D-dimensional space-time and $\tau^\alpha = (\tau, \sigma)$ are the coordinates on the world-sheet. Following figure 1, a convenient gauge fixing choice is $t = X^0 = \tau, x = X^1 = \sigma$. Take a finite time extend $\int d\tau = T$. Note from the figure that, on the string embedding r is a function of $\sigma = x = X^1$. Calculate the gauge fixed action and show that:

$$S_{NG} = \frac{T}{2\pi\alpha'} \int dx \sqrt{f^2(r(x)) + g^2(r(x))(\partial_x r)^2} \,, \quad r' \equiv \partial_x r \,, \tag{4}$$

with

$$f^2 = G_{00} G_{xx} \,, \quad g^2 = G_{00} G_{rr} \,, \tag{5}$$

clearly

b. [**7 points**] Consider the NG Lagrangian $\mathcal{L} = \sqrt{f^2(r(x)) + g^2(r(x))(r')^2}$. Find the corresponding Hamiltonian density and show that it can be brought into the following form:

$$\mathcal{H} = -\frac{f^2}{\mathcal{L}} \,. \tag{6}$$

c. [**7 points**] Note that the Lagrangian above does not depend on $x = \sigma$ explicitly. What does this imply for the Hamiltonian density? (Hint: think of the analogous classical mechanical problem where the Lagrangian does not depend on time. One can then fix the Hamiltonian density with some value for x. You are allowed to assume that there exists a $x = x_0$ such that $r(x_0)$ takes the minimal value of $r(x)$.). Use this observation to derive the equations of motion for the string:

$$\frac{dr}{dx} = \pm \frac{f(r)}{g(r)} \frac{\sqrt{f^2(r) - f^2(r_0)}}{f(r_0)} \,, \tag{7}$$

with $f(r_0) = f_0$ the value of f at the tip of the string, see fig.1.

d. [**8 points**] Consider now the specific case of the Anti-de-Sitter geometry,

$$ds^2 = \frac{1}{r^2} \left(-dt^2 + dr^2 + \delta_{ij} dx^i dx^j \right) \,,$$

where $r = r_1 = 0$ corresponds to the boundary. Write down the equation of motion for the string in this case. Find a formula that implicitly determines r_0 in terms of L. What happens to r_0 if you increase L? Does it move towards the boundary or deep

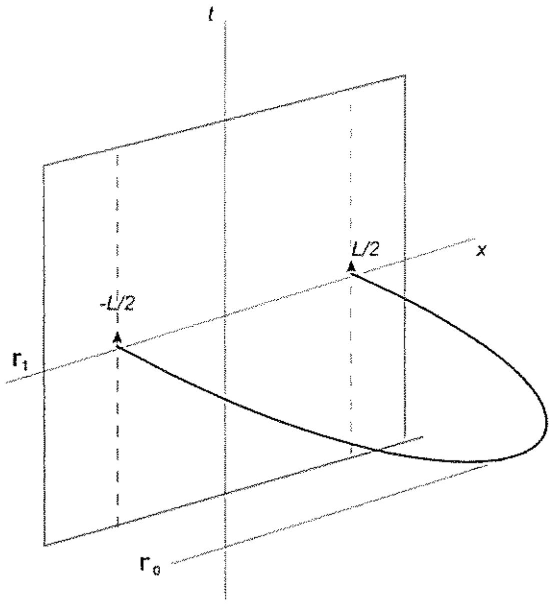

Figure 1: Typical form of a Nambu-Goto string with boundary conditions that the end-points of the string touches the boundary at r_1 at points $x = \pm L/2$. Here the string is hanging down the r direction.

- Are the states you found corresponding to space-time bosons or fermions?

b. [**10 points**] Construct the states in the R sector of the open superstring spectrum at mass levels $\alpha' M^2 = 0$, $\alpha' M^2 = +2$ and $\alpha' M^2 = +3$. Consider both of the possibilities with the Ramond ground state left handed and right handed, i.e. 8_c (negative chirality) and 8_s (positive chirality) in the notation of the notes. Answer the same questions as above.

c. [**5 points**] The GSO operator in the R sector is defined by $G_R = \Gamma^9(-1)^F$, where $\Gamma^9 = \pm 1$ measures the chirality of spinors in eight dimensions where $F = \sum_{r>0} b^i_{-r} b^i_r$. Which states you found above survive the GSO projection $G_R = +1$? Which ones survive $G_R = -1$?

d. [**Bonus: 5 points**] Is there a graviton state among the states you constructed above? If yes, which one? If no, where is the graviton?

4. Nambu-Goto string on curved space-time (30 points + 5 bonus)

In this exercise you will calculate the on-shell Nambu-Goto action of a classical string embedded in a nontrivial curved space-time geometry in D dimensions.

a. [**8 points**] Consider a general space with the metric $ds^2 = G_{\mu\nu} dx^\mu dx^\nu =$

interior of the geometry?

e. **[Bonus: 5 points]** Finally consider another specific case of the Anti-de-Sitter black-hole geometry,

$$ds^2 = \frac{1}{r^2}\left(-V(r)dt^2 + \frac{dr^2}{V(r)} + \delta_{ij}dx^i dx^j\right),$$

At

where $V(r) = 1 - \left(\frac{r}{r_h}\right)^{D-1}$ is called the blackening factor. The locus it vanishes at $r = r_h$ corresponds to the event horizon of the black-hole. What happens to the string solution when r_0 touches the horizon? Suppose that $r_0 = r_h$ when $L = L_h$. Based on your finding in **d** how do you expect the string solution will look like when $L > L_h$?