EXAMINER: DR. THOMAS W. GRIMM
DATE: 11/04/2019
TIME: 09:00 - 12:00

UTRECHT UNIVERSITY
MIDTERM EXAM

Midterm exam for String Theory

- Write your **name and student number on every sheet**.
- There are five problems. Write your answers to the individual problems on different sheets.
- No lecture notes, books or anything else is allowed. In particular, don't use a pencil for your answers.
- Make sure that your **answers are understandable and readable**. In doubt, explain with a short comment on what you're doing.
- You can reach a total of **48 points**.
- On the last page you can find some formulas which might be useful.

Problem 1: Short questions [10 points]

In this problem we will ask you some basic questions concerning the lecture. You should give **short** answers. Don't lose too much time on Problem 1.

(i) Give reasons why one might want to study string theory? Why is gravity at high energies an exception compared to the other fundamental forces in nature?

(ii) State the Nambu-Goto and the Polyakov action. What is the connection of these two actions?

(iii) Name the symmetries of the Polyakov action.

(iv) What are the consequences of the local symmetries for the energy-momentum tensor $T_{\alpha\beta}$?

(v) Give the Virasoro constraints for the classical and quantized closed bosonic string theory.

(vi) Why does one have to impose the Virasoro constraints? What is their origin?

(vii) List the space-time fields corresponding to the level one states of the closed bosonic string and the open bosonic string on a single Dp-brane extending along $X^i, i = 0, \ldots, p$. The states have eigenvalues 1 under the level operator N.

(viii) How many scalars are included on the worldsheet in critical bosonic string theory and critical superstring theory?

(ix) What is the definition of a Weyl transformation? Explain its difference compared to a conformal transformation.

(x) What is special about the algebra of infinitesimal conformal transformations in two dimensions compared to the higher-dimensional cases?

Problem 2: Classical string solution [10 points]

Consider the following configuration of a **classical** open string,

$$\begin{aligned} X^0 &= B\tau, \\ X^1 &= B\cos(\tau)\cos(\sigma), \\ X^2 &= B\sin(\tau)\cos(\sigma), \\ X^i &= 0\,, \qquad i > 2\,, \end{aligned} \tag{1}$$

where $0 \leq \sigma \leq \pi, B > 0$. In the following you may assume that the world-sheet metric is fixed to flat gauge, i.e. $h_{\alpha\beta} = \eta_{\alpha\beta}$.

(i) Show that this configuration describes a solution to the equations of motion (following from the Polyakov action) for the field $X^\mu(\tau, \sigma)$ corresponding to an open string with Neumann-Neumann boundary conditions along all directions.

(ii) Consider a point on this string at fixed σ. Calculate the speed of this point via

$$v = \sqrt{\left(\frac{\mathrm{d}X^1}{\mathrm{d}X^0}\right)^2 + \left(\frac{\mathrm{d}X^2}{\mathrm{d}X^0}\right)^2}\,. \tag{2}$$

What is the value of the speed of the endpoints of this string (recall that we work in units $c = 1$)?

(iii) Consider the conserved charges

$$\begin{aligned} P^\mu &= \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\, \partial_\tau X^\mu\,, \\ J^{\mu\nu} &= \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\, \left(X^\mu\partial_\tau X^\nu - X^\nu\partial_\tau X^\mu\right). \end{aligned} \tag{3}$$

Compute the energy $E = P^0$ and angular momentum $J = J^{12}$ of the string and show that

$$\frac{E^2}{|J|} = \frac{1}{\alpha'}\,. \tag{4}$$

(iv) The energy-momentum tensor of the Polyakov action is in general given by

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}\, h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu\,. \tag{5}$$

Show explicitly that the solution (1) satisfies the constraint $T_{\alpha\beta} = 0$.

Problem 3: Closed string states [8 points]

In this problem we work with closed bosonic strings in the critical dimension and in lightcone quantization, where the spacetime coordinates are indexed by $i = +, -, 1, 2, \ldots, D-2$. We choose units such that the mass operator is dimensionless, given by

$$M^2 = 4\left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n - 1\right). \tag{6}$$

(i) Consider the states

$$\alpha^i_{-p}\tilde{\alpha}^j_{-q}\tilde{\alpha}^k_{-r}\,|0;p^\mu\rangle\,, \tag{7}$$

where the mode numbers p, q, r are positive integers satisfying $p+q+r \leq 4$. What are the possible values of the mode numbers p, q, r for (7) to be physical? What are their masses? It is enough to give arguments based on level-counting.

(ii) Determine the independent components for every physical state you find in (i). You may express your answer in terms of D. Explain your counting briefly.

(iii) What is the highest possible spin of the state (7)?

(iv) The mass formula (6) is dimensionless in our chosen units. Now we choose mass (kg), length (m), and time (s) as basic units. Reinstall the string tension T, Planck constant $\hbar$, and the speed of light c in the mass formula such that M has the dimension of mass. You may ignore numerical factors other than $T, \hbar$, and c.

Hint: The Planck constant $\hbar$ has unit $\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}$.

Problem 4: Open string propagator [8 points]

Recall the mode expansion of the open string with (NN) boundary conditions

$$\text{(NN):} \qquad X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\alpha_n^\mu \, \mathrm{e}^{-in\tau}\cos(n\sigma). \tag{8}$$

The propagator is as usual defined by

$$\langle X^\mu(\tau,\sigma)\, X^\nu(\tau',\sigma')\rangle = \mathcal{T}\big[X^\mu(\tau,\sigma)\, X^\nu(\tau',\sigma')\big] - :X^\mu(\tau,\sigma)\, X^\nu(\tau',\sigma'): \;. \tag{9}$$

The time ordering operator $\mathcal{T}$ is as usual defined by

$$\mathcal{T}\big[A(\tau)\, B(\tau')\big] = \begin{cases} A(\tau)\, B(\tau')\,, & \text{if } \tau > \tau', \\ B(\tau')\, A(\tau)\,, & \text{if } \tau < \tau'. \end{cases} \tag{10}$$

(i) Introduce the new coordinates $(z,\bar{z}) \in S^1 \times S^1$ with $z = \mathrm{e}^{i\sigma^-}$ and $\bar{z} = \mathrm{e}^{i\sigma^+}$ $(\sigma^\pm = \tau \pm \sigma)$[1]. Express the mode expansions (8) in terms of the new variables $(z,\bar{z})$.

(ii) Show that the open string propagator with NN ([redacted]) boundary conditions is given by

$$\langle X^\mu(z,\bar{z})\, X^\nu(w,\bar{w})\rangle = -\alpha'\eta^{\mu\nu}\left(\log|z-w| + \log|z-\bar{w}|\right). \tag{11}$$

Hint: You may assume without loss of generality that

$$\mathcal{T}\big[X^\mu(\tau,\sigma)\, X^\nu(\tau',\sigma')\big] = X^\mu(\tau,\sigma)\, X^\nu(\tau',\sigma')\,.$$

[1] The variables z and $\bar{z}$ are **not** related by complex conjugation.

Problem 5: Virasoro algebra [12 points]

In this problem we derive the quantum Virasoro commutators $[L_n, L_m]$ from self-consistency arguments. For simplicity, we work with closed bosonic strings in covariant quantisation, and focus only on the right-moving sector, i.e., we ignore $\tilde{L}_n$. Recall that the classical Witt algebra operators l_n satisfy the following commutation relation

$$[l_m, l_n] = (m-n)l_{m+n}, \tag{12}$$

for m, n integers. Upon quantisation, we define the operators L_n in the quantum Virasoro algebra to be the classical l_n with a normal ordering prescription. This normal ordering modifies the commutator into

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}, \tag{13}$$

where m, n are integers and $A(m)$ is a number depending on m.

Our task is to find this function $A(m)$.

(i) Explain briefly the appearance of $\delta_{m+n,0}$, i.e. discuss why there is a modification at the quantum level only for commutators $[L_n, L_{-n}]$.

(ii) Determine the value of $A(0)$. Show that if $A(1) \neq 0$, then it is possible to redefine L_0 by adding a constant, such that $A(1) = 0$. Show that $A(-m) = -A(m)$ for all m.

(iii) Assume now $A(1) = 0$. Use the Jacobi identity to derive a recursion relation among $A(m)$'s. And then solve for $A(m)$ with $m > 2$ in terms of $A(2)$. You should find $A(m) = \frac{(m^3-m)A(2)}{6}$.

Hint: The Jacobi identity is given by

$$[L_m, [L_n, L_r]] + [L_n, [L_r, L_m]] + [L_r, [L_m, L_n]] = 0. \tag{14}$$

Try applying the Jacobi identity with $m + n + r = 0$.

(iv) From the definition of L_n in terms of the mode operators α_k^μ, deduce that we have indeed $A(1) = 0$. Furthermore, determine the value of $A(2)$. These can be done by computing the expectation value $\langle 0; 0|[L_m, L_{-m}]|0; 0\rangle$ for $m = 1$ and 2, where $|0; 0\rangle$ is the ground state with a vanishing momentum $p^\mu = 0$. You should find $A(2) = \frac{D}{2}$, where D is the space-time dimension.

Hint: For $m = 2$ a useful intermediate result that you should find is

$$\langle 0; 0|[L_2, L_{-2}]|0; 0\rangle = \langle 0; 0|\alpha_1 \cdot \alpha_1\, \alpha_{-1} \cdot \alpha_{-1}|0; 0\rangle\,.$$

Closed string mode expansions:

The mode expansions for the closed strings $X^\mu = X^\mu_L + X^\mu_R$ is given by

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+},$$
$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-},$$

where the worldsheet light cone coordinates are $\sigma^\pm = \tau \pm \sigma$. We furthermore identify

$$\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}\,p^\mu.$$

Some useful formulas:

Taylor series for $\log(1+x)$:

$$\log(1+x) = \sum_{k=1}^{\infty}\frac{1}{k}(-1)^{k+1}x^k$$

Commutation relations:

$$[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}$$

The quantum Virasoro operators:

$$L_n = \frac{1}{2} : \sum_{k\in\mathbb{Z}} \alpha_{n-k}\cdot\alpha_k :$$