Wat is Wiskunde Re-Exam A, 21/12/2009, English

Voor de Nederlandse tekst van dit tentamen zie ommezijde.

- On each sheet of paper you hand in write your name and student number
- Each problem counts for 20 points, leading to a maximum of 100 points
- Do not provide just final answers. Prove and motivate your arguments!
- The use of computer, calculator, lecture notes, or books is not allowed

Problem A) Determine which of the following three expressions are logically equivalent

$1: \ (P \wedge (\neg Q)) \Rightarrow (R \vee Q) \qquad 2: \ (\neg P) \vee Q \vee R \qquad 3: \ (\neg P) \vee (\neg Q) \vee R$

Problem B) Prove by induction that for every integer $n > 0$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Problem C)

(1) Prove that for every three sets A, B and C holds

$$((A - B) \cup (B - A)) \cap C = ((A \cap C) \cup (B \cap C)) - (A \cap B)$$

(2) Show that the equality

$$A - (B - (C - D)) = ((A - B) - C) - D$$

does not necessarily hold for all sets A, B, C and D.

Problem D)

(1) Consider the set $\mathbb{Z}$ of integers and the relation R given by: for $x, y \in \mathbb{Z}$ holds xRy precisely when the number $|x - y| + 1$ is either 1 or a prime number. Prove that R is not an equivalence relation.

(2) Consider the set $\mathbb{R} \times \mathbb{R} = \{(a, b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$ of pairs of real numbers and the relation S given by: for $(x_1, x_2), (y_1, y_2) \in \mathbb{R} \times \mathbb{R}$ holds $(x_1, x_2)S(y_1, y_2)$ exactly when $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Prove that S is an equivalence relation.

(3) Consider the same relation S as in (2) and the equivalence classes $A_1 = [(0,0)], A_2 = [(0,2)], A_3 = [(\sqrt{2}, \sqrt{2})]$. Which of A_1, A_2, A_3 are equal? Which of A_1, A_2, A_3 are finite sets.

Problem E) For each of the following statements decide if it is true or false. Give a short argument to support your answer.

(1) If R is an equivalence relation on a finite set A then the number of distinct equivalence classes is either 1 or a prime number.

(2) Let B_1, B_2, B_3 be three sets. If $B_1 \cap B_2 \neq \emptyset$ and $B_1 \cap B_3 \neq \emptyset$ and $B_2 \cap B_3 \neq \emptyset$ then $B_1 \cap B_2 \cap B_3 \neq \emptyset$.

(3) Let A be a set. There exists at least one equivalence relation R on A.

Wat is Wiskunde herkansing A, 21/12/2009, Nederlands

For the English text of this exam see the back of this page.

- Schrijf op elk blad dat je inlevert je naam en studentnummer.
- Elk van de vijf opgaven telt voor 20 punten.
- Geef niet alleen eindantwoorden, maar laat ook duidelijk zien hoe je tot je antwoord komt.
- Gebruik van een computer, rekenmachine, aantekeningen of boeken tijdens dit tentamen is niet toegestaan

Opgave A) Bepaal welke van de volgende drie beweringen logisch equivalent zijn

$1: \ (P \wedge (\neg Q)) \Rightarrow (R \vee Q) \qquad 2: \ (\neg P) \vee Q \vee R \qquad 3: \ (\neg P) \vee (\neg Q) \vee R$

Opgave B) Bewijs met inductie dat voor elk geheel getal $n > 0$ geldt

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Opgave C)

(1) Bewijs dat voor elk drietal verzamelingen A, B, C geldt

$$((A - B) \cup (B - A)) \cap C = ((A \cap C) \cup (B \cap C)) - (A \cap B)$$

(2) Bewijs dat de gelijkheid

$$A - (B - (C - D)) = ((A - B) - C) - D$$

niet hoeft te gelden voor elk viertal verzamelingen A, B, C, D.

Opgave D)

(1) Neem op de verzameling $\mathbb{Z}$ van de gehele getallen de relatie R gegeven door: voor $x, y \in \mathbb{Z}$ geldt xRy precies dan als $|x - y| + 1$ gelijk is aan 1 of een priem getal. Bewijs dat R niet een equivalentie relatie is.

(2) We beschouwen op de verzameling $\mathbb{R} \times \mathbb{R} = \{(a, b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$ van paren van reële getallen de relatie S gegeven door: voor $(x_1, x_2), (y_1, y_2) \in \mathbb{R} \times \mathbb{R}$ geldt $(x_1, x_2)S(y_1, y_2)$ precies dan als $x_1^2 + x_2^2 = y_1^2 + y_2^2$.
Bewijs dat S een equivalentie relatie is.

(3) Neem dezelfde relatie S als in (2) en beschouw de equivalentieklassen $A_1 = [(0, 0)], A_2 = [(0, 2)], A_3 = [(\sqrt{2}, \sqrt{2})]$. Welke van de equivalentieklassen A_1, A_2, A_3 zijn gelijk? Welke van de equivalentieklassen A_1, A_2, A_3 is een eindige verzameling?

Opgave E) Geef voor elk van de onderstaande beweringen aan of hij juist of onjuist is. Geef een kort argument ter ondersteuning van je antwoord.

(1) Als R een equivalentie relatie op een eindige verzameling A is dan is het aantal verschillende equivalentieklassen gelijk aan 1 of een priemgetal.

(2) Zij B_1, B_2, B_3 een drietal verzamelingen. Als $B_1 \cap B_2 \neq \emptyset$ en $B_1 \cap B_3 \neq \emptyset$ en $B_2 \cap B_3 \neq \emptyset$ dan geldt $B_1 \cap B_2 \cap B_3 \neq \emptyset$.

(3) Zij A een verzameling. Er bestaat minstens 1 equivalentie relatie op A.

Wat is Wiskunde Re-Exam A - Solutions

Problem A) Determine which of the following three expressions are logically equivalent

$$1: \ (P \wedge (\neg Q)) \Rightarrow (R \vee Q) \qquad 2: \ (\neg P) \vee Q \vee R \qquad 3: \ (\neg P) \vee (\neg Q) \vee R$$

Solution: By calculating and inspecting the truth tables for each of the three expressions one verifies that only expressions 1 and 2 are equivalent.

Problem B) Prove by induction that for every integer $n > 0$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Solution: By induction on n. The induction basis is the case $n = 1$ in which case the left hand side becomes $1^3 = 1$ and the right hand side is equal to $\frac{1^2(1+1)^2}{4} = 1$. This establishes the induction basis. For the induction step assume the equality holds for $n = k$ and we now set out to prove that it also holds for $k+1$. The left hand side is in this case equal to

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$

which by grouping together the first k summands and using the induction hypothesis is seen to be equal to

$$\frac{k^2(k+1)^2}{4} + (k+1)^3.$$

We wish to show that this is equal to the right hand side of the original equation where $n = k+1$. That right hand side is:

$$\frac{(k+1)^2(k+2)^2}{4}$$

and we thus need to establish that

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}.$$

Simplifying the left hand side we have

$$\frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k^2+4k+1)(k+1)^2}{4} = \frac{(k+2)^2(k+1)^2}{4}$$

as desired. This proves the induction step and thus the equality for all natural numbers $n > 0$ by the induction principal.

Problem C)

(1) Prove that for every three sets A, B and C holds

$$((A - B) \cup (B - A)) \cap C = ((A \cap C) \cup (B \cap C)) - (A \cap B)$$

(2) Show that the equality

$$A - (B - (C - D)) = ((A - B) - C) - D$$

does not necessarily hold for all sets A, B, C and D.

Solution: 1) Let $x \in ((A-B)\cup(B-A))\cap C$. Then $x \in (A-B)\cup(B-A)$ and $x \in C$. It thus follows that either $x \in A-B$ or $x \in B-A$. Due to symmetry considerations we may assume without loss of generality that $x \in A-B$. Thus we have that $x \in A-B$ and $x \in C$ which means that $x \in A$ and $x \notin B$ and $x \in C$. Since $x \in A$ and $x \in C$ it follows that $x \in A\cap C$ and thus also that $x \in ((A\cap C)\cup(B\cap C))$. Since $x \notin B$ it follows also that $x \notin A\cap B$ and thus we conclude that $x \in ((A\cap C)\cup(B\cap C))$ and $x \notin A\cap B$ which means that $x \in ((A\cap C)\cup(B\cap C)) - (A\cap B)$. This argument proves that $((A-B)\cup(B-A)))\cap C \subseteq ((A\cap C)\cup(B\cap C)) - (A\cap B)$. For the other direction let $y \in ((A\cap C)\cup(B\cap C)) - (A\cap B)$. Then $y \in ((A\cap C)\cup(B\cap C))$ and $y \notin A\cap B$. Thus either $y \in A\cap C$ or $y \in B\cap C$. Again due to symmetry we may, without loss of generality, assume that $y \in A\cap C$. We thus have that $y \in A\cap C$ and $y \notin A\cap B$. This means that $y \in A$ and $y \in C$ and either $y \notin A$ or $y \notin B$. Since $y \in A$ we conclude that $y \notin B$. We thus have that $y \in A$ and $y \in C$ and $y \notin B$. Since $y \in A$ and $y \notin B$ it follows that $y \in A-B$ and thus also that $y \in ((A-B)\cup(B-A))$. Since $y \in C$ we conlude that $y \in ((A-B)\cup(B-A))\cap C$. This argument now established that $((A\cap C)\cup(B\cap C)) - (A-B) \subseteq ((A-B)\cup(B-A))\cap C$. The two arguments established the required equality.

2) We provide the counter example: $A = \{1\}, B = \{1,2\}, C = \{1,2\}, D = \emptyset$. Then

$$A - (B - (C - D)) = A - (B - C) = A - \emptyset = A$$

while

$$((A - B) - C) - D = (\emptyset - C) - D = \emptyset - D = \emptyset$$

and since $A \neq \emptyset$ our counter example is valid.

Problem D)

(1) Consider the set $\mathbb{Z}$ of integers and the relation R given by: for $x, y \in \mathbb{Z}$ holds xRy precisely when the number $|x-y|+1$ is either 1 or a prime number. Prove that R is not an equivalence relation.

(2) Consider the set $\mathbb{R}\times\mathbb{R} = \{(a,b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$ of pairs of real numbers and the relation S given by: for $(x_1,x_2),(y_1,y_2) \in \mathbb{R}\times\mathbb{R}$ holds $(x_1,x_2)S(y_1,y_2)$ exactly when $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Prove that S is an equivalence relation.

(3) Consider the same relation S as in (2) and the equivalence classes $A_1 = [(0,0)], A_2 = [(0,2)], A_3 = [(\sqrt{2},\sqrt{2})]$. Which of A_1, A_2, A_3 are equal? Which of A_1, A_2, A_3 are finite sets.

Solution; 1) To show that R is not an equivalence relation we show it is not transitive. Let $x = 5, y = 3, z = 2$. xRy since $|x-y|+1 = 3$ is prime. yRz also holds since

$|y-z|+1=2$, a prime number. But xRz does not hold since $|x-z|+1=4$ is not prime and not equal to 1.
2) We need to show that S is reflexive, symmetric, and transitive. Let $(x,y)\in\mathbb{R}\times\mathbb{R}$. Then $(x,y)S(x,y)$ holds since $x^2+y^2=x^2+y^2$. Thus S is reflexive. Now let $(x_1,x_2),(y_1,y_2)\in\mathbb{R}\times\mathbb{R}$ and assume $(x_1,x_2)S(y_1,y_2)$. That means that $x_1^2+x_2^2=y_1^2+y_2^2$. Clearly then $y_1^2+y_2^2=x_1^2+x_2^2$ which implies $(y_1,y_2)S(x_1,x_2)$ and thus that S is symmetric. Now let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in\mathbb{R}\times\mathbb{R}$ such that $(x_1,x_2)S(y_1,y_2)$ and $(y_1,y_2)S(z_1,z_2)$. That means that $x_1^2+x_2^2=y_1^2+y_2^2$ and $y_1^2+y_2^2=z_1^2+z_2^2$ which clearly implies that $x_1^2+x_2^2=z_1^2+z_2^2$. That is: $(x_1,x_2)S(z_1,z_2)$ holds which means S is transitive. We have thus shown that S is an equivalence relation.
3) A property of equivalence classes is that two such, $[x]$ and $[y]$ are equal as sets if, and only if, xSy. Thus to see which of the three classes given are equal we need to check when the representatives are related in S. Now since $0^2+2^2=4=\sqrt{2}^2+\sqrt{2}^2$ it follows that $(0,2)S(\sqrt{2},\sqrt{2})$ and thus that $A_2=A_3$. And since $0^2+0^2=0\neq 4$ it follows that $(0,0)$ is not related to $(0,2)$ nor to $(\sqrt{2},\sqrt{2})$ and thus that $A_1\neq A_2$ and $A_1\neq A_3$. To see which are finite sets we look at the definition of the equivalence class: $A_1=[(0,0)]=\{(x,y)\in\mathbb{R}\times\mathbb{R}\mid x^2+y^2=0^2+0^2=0\}$. The only solution to this equation is $x=y=0$ and thus $[(0,0)]=\{(0,0)\}$ which is a finite set. Since $A_2=A_3$ we just need to determine one of the two. We have $A_2=\{(x,y)\in\mathbb{R}\times\mathbb{R}\mid x^2+y^2=0^2+2^2=4\}$. The solutions to this equation are all the points in the plane that lie on the circle of radius 2 about the origin. Thus A_2 (and A_3) is infinite.

Problem E) For each of the following statements decide if it is true or false. Give a short argument to support your answer.

(1) If R is an equivalence relation on a finite set A then the number of distinct equivalence classes is either 1 or a prime number.
(2) Let B_1,B_2,B_3 be three sets. If $B_1\cap B_2\neq\emptyset$ and $B_1\cap B_3\neq\emptyset$ and $B_2\cap B_3\neq\emptyset$ then $B_1\cap B_2\cap B_3\neq\emptyset$.
(3) Let A be a set. There exists at least one equivalence relation R on A.

Solution: 1) This is not true. In fact the number of equivalence classes can be any number at all. For example let A be a set with n elements for some natural number n. The relation R on A in which for $a,b\in A$ holds aRb if, and only if, $a=b$ is easily seen to be an equivalence relation and the number of equivalence classes is equal to n. For a more concrete counter example: We know that equality modulo 4 is an equivalence relation on $\mathbb{Z}$ and we know that there are then 4 equivalence relations.
2) This in not true. Let $B_1=\{2.3\},B_2=\{1,2\},B_3=\{1,3\}$. Then $B_1\cap B_2=\{2\}$ and $B_1\cap B_3=\{3\}$ and $B_2\cap B_3=\{1\}$. However, $B_1\cap B_2\cap B_3=\emptyset$.
3) This is true. For example we can choose for A the relation $S=A\times A$, that is for any $a,b\in A$ holds aSb. It is then trivial that S is an equivalence relation. (one can also take the construction of part 1) of this problem.