Wat is Wiskunde Exam B, 17-01-2011, English

- Voor de Nederlandse tekst van dit tentamen zie ommezijde.
- Write the solution to each problem on a separate sheet of paper
- On each sheet of paper you hand in write your name and student number
- Each problem counts for 20 points, leading to a maximum of 100 points
- Do not provide just final answers. Prove and motivate your arguments!
- The use of computer, calculator, lecture notes, or books is not allowed. A personal A4 is allowed.

Problem A) Consider the real valued assignment $f(x) = \frac{x}{x-1}$ and let $A \subseteq \mathbb{R}$ be the largest subset of the real numbers on which the function is well-defined.

(1) Find the set A and prove that the image $\{f(a) \mid a \in A\} = A$.
(2) Prove that $f : A \to A$ is bijective and find its inverse.
(3) Write f^n for the composition of f with itself n times. Give an explicit formula for $f^3(x)$.
(4) Calculate $f^{1000}(0)$.

Problem B) (new sheet!)

(1) State (without proof) the Cantor-Schroeder-Bernstein Theorem.
(2) Let $S = [-1, 1] \times [-1, 1]$ and $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Prove that $|S| = |C|$ (hint: think of these sets geometrically).
(3) Prove that the set $X = \{A \subseteq \mathbb{N} \mid |A| < \omega\}$, that is the set of all finite subsets of $\mathbb{N}$, is countably infinite.

Problem C) (new sheet!)

(1) Let $d = gcd(173, 2011)$. Use the Euclidean Algorithm to find d and write d as $x \cdot 2011 + y \cdot 173$ where x and y are integers.
(2) Let a, b be two positive natural numbers and let $gcd(a, b) = c$. Prove that $gcd(a^2, b) \leq c^2$.

Problem D) (new sheet!)

(1) Let $(G, *, e)$ be a group. Prove that for every $g, h, k \in G$ holds that $(g * h * k)^{-1} = k^{-1} * h^{-1} * g^{-1}$.
(2) Let $A = \{1, 2, 3\}$. For each element x in the group $Symm(A)$, the group of bijections from A to A, find the least natural number n_x such that in $Symm(A)$ holds $x * x * \cdots * x$ (n_x times) is the unit element.

Problem E) (new sheet!) For each of the following statements decide if it is true or false. Give a short argument to support your answer.

(1) Let A be a set and $f : A \to A$ a function. If $f \circ f \circ f \circ f \circ f$ is invertible then f is invertible.
(2) There exists a set X and a subset $Y \subseteq X$ such that $X \neq Y$ and $|X| = |Y|$.
(3) For any natural numbers a, b holds that if $a \neq b$ then $gcd(a^b, b^a) = 1$.
(4) Let $(G, *, e)$ be a group. For any two elements $g, h \in G$ holds that $(g * h)^{-1} = g^{-1} * h^{-1}$.

WAT IS WISKUNDE TENTAMEN B, 17-01-2011, NEDERLANDS

- Please turn over for the English text of this exam.
- Schrijf de uitwerking van iedere opgave met een apart vel papier
- Schrijf op ieder vel paier dat je inlevert je naam en studentnummer
- Iedere opgave is 20 punten waard. Je kunt maximaal 100 puten halen.
- Geef niet alleen uitkomsten. Bewijs en motiveer je antwoorden!
- Het gebruik van een computer, rekenmachine, dictaat of boeken is niet toegestaan. Je mag een persoonlijk A4 gebruiken.

Opgave A) Beschouw het reëelwaardige voorschrift $f(x) = \frac{x}{x-1}$ en laat $A \subseteq \mathbb{R}$ de grootste deelverzameling van de reële getallen zijn zodat de functie goed gedefinieerd is.

(1) Bepaal de verzameling A en bewijs dat $\{f(a) \mid a \in A\} = A$.
(2) Bewijs dat $f : A \to A$ bijectief is en bepaal de inverse.
(3) Schrijf f^n voor de n-voudige samenstelling van f met zichzelf. Geef een expliciete formule voor $f^3(x)$.
(4) Bereken $f^{1000}(0)$.

Opgave B) (nieuw vel!)

(1) Formuleer (zonder bewijs) de stelling van Cantor-Schroeder-Bernstein.
(2) Laat $S = [-1, 1] \times [-1, 1]$ en $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Bewijs dat $|S| = |C|$ (hint: interpreteer deze verzamelingen meetkundig).
(3) Bewijs dat de verzameling $X = \{A \subseteq \mathbb{N} \mid |A| < \omega\}$, de verzameling van alle eindige deelverzamelingen van $\mathbb{N}$, aftelbaar oneindig is.

Opgave C) (nieuw vel!)

(1) Laat $d = ggd(173, 2011)$. Gebruik het algoritme van Euclides om d te bepalen en schrijf d als $x \cdot 2011 + y \cdot 173$ waarbij x en y gehele getallen zijn.
(2) Laat a, b twee positieve natuurlijke getallen zijn, en laat $ggd(a, b) = c$. Bewijs dat $ggd(a^2, b) \leq c^2$.

Opgave D) (nieuw vel!)

(1) Laat $(G, *, e)$ een groep zijn. Bewijs dat voor alle $g, h, k \in G$ geldt dat $(g * h * k)^{-1} = k^{-1} * h^{-1} * g^{-1}$.
(2) Laat $A = \{1, 2, 3\}$. Bepaal voor ieder element x van de groep $Symm(A)$, de groep van bijecties van A naar A, het kleinste natuurlijk getal n_x zodat in $Symm(A)$ geldt dat $x * x * \cdots * x$ (n_x keer) het eenheidselement is.

Opgave E) (nieuw vel!) Bepaal voor iedere van de volgende beweringen of hij juist of onjuist is. Geef een kort argument om je antwoord te ondersteunen.

(1) Laat A een verzameling zijn en $f : A \to A$ een functie. Als $f \circ f \circ f \circ f \circ f$ inverteerbaar is, dan is f inverteerbaar.
(2) Er bestaan een verzameling X en een deelverzameling $Y \subseteq X$ zodat $X \neq Y$ en $|X| = |Y|$.
(3) Voor alle natuurlijke getallen a, b geldt dat als $a \neq b$ dan $ggd(a^b, b^a) = 1$.
(4) Laat $(G, *, e)$ een groep zijn. Voor ieder tweetal elementen $g, h \in G$ geldt dat $(g * h)^{-1} = g^{-1} * h^{-1}$.

SOLUTIONS

Problem A) Consider the real valued assignment $f(x) = \frac{x}{x-1}$ and let $A \subseteq \mathbb{R}$ be the largest subset of the real numbers on which the function is well-defined.

1. Find the set A and prove that the image $\{f(a) \mid a \in A\} = A$.
2. Prove that $f : A \to A$ is bijective and find its inverse.
3. Write f^n for the composition of f with itself n times. Give an explicit formula for $f^3(x)$.
4. Calculate $f^{1000}(0)$.

Solution

1. The function is well defined unless $x - 1 = 0$. Thus the domain of definition is $A = \{x \in \mathbb{R} \mid x \neq 1\}$. To show that the image of f is A as well we need to find for each $y \in A$ some $x \in A$ for which $f(x) = y$. Let $y \in A$ thus. We solve $f(x) = y$ for x:
$$\frac{x}{x-1} = y$$
which implies: $x = xy - y$ which leads to $y = xy - x = x(y-1)$ and finaly, since $y \neq 1$, $x = \frac{y}{y-1}$. Thus a solution exists and so the image is A.
2. We prove that f is bijective by finding its inverse. In fact, in 1 above, we found that the inverse is $g(x) = \frac{x}{x-1}$. We verify that again by calculating
$$f(g(x)) = \frac{g(x)}{g(x)-1} = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{\frac{x}{x-1}}{\frac{1}{x-1}} = \frac{x}{x-1} \cdot (x-1) = x.$$
Note that it is clear that $g(f(x)) = x$ since $g = f$.
3. This can be done by direct calculation or, simpler, as follows (remembering that $f^{-1} = f = g$, we compute:
$$f^3(x) = f \circ f \circ f(x) = f \circ g \circ f(x) = (f \circ g)(f(x)) = f(x).$$
Thus the explicite formula is $f^3(x) = \frac{x}{x-1}$.
4. Note that $f(0) = \frac{0}{0-1} = 0$. Thus 0 is a fixed point of f and so $f^n(0) = 0$ for all natural numbers n. The case in question is $n = 1000$. So: $f^{1000}(0) = 0$. (There are other ways one can solve this problem.)

Problem B) (<u>new sheet</u>!)

1. State (without proof) the Cantor-Schroeder-Bernstein Theorem.
2. Let $S = [-1,1] \times [-1,1]$ and $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Prove that $|S| = |C|$ (hint: think of these sets geometrically).
3. Prove that the set $X = \{A \subseteq \mathbb{N} \mid |A| < \omega\}$, that is the set of all finite subsets of $\mathbb{N}$, is countably infinite.

Solution:

(1) The theorem of Cantor-Schroeder-Bernstein states that given any two sets A and B if there is an injection $f : A \to B$ and an injection $g : B \to A$ then there exists a bijection $h : A \to B$.

(2) According to the Cantor-Schroeder-Bernstein Theorem it is sufficient to find injections $f : C \to S$ and $g : S \to C$. Geometrically, it is clear that C (the circle of radius 1) is contained in S (the square with side equal to 1). So we define $f : C \to S$ by $f(x, y) = (x, y)$. Let us verify that f indeed has S as codomain. Let $(x, y) \in C$, thus $x^2 + y^2 \leq 1$. We need to show that both $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. This is indeed the case since if, without loss of generality, $|x| > 1$ then $x^2 > 1$ which would make $x^2 + y^2 \leq 1$ impossible (since $y^2 \geq 0$). It is obvious that f is injective. To find $g : S \to C$ we need to shrink the square by a factor to get it to fit inside the circle. Let us choose a factor of $\alpha = \frac{1}{2}$. Now we define $g(x, y) = (\alpha x, \alpha y)$. We need to verify that indeed the codomain of g is C. So let $(x, y) \in S$ which means that $|x| \leq 1$ and $|y| \leq 1$. We calculate:

$$(\alpha x)^2 + (\alpha y)^2 = \alpha^2(x^2 + y^2) = \alpha^2(|x|^2 + |y|^2) \leq \alpha^2(1 + 1) = \frac{1}{2^2} \cdot 2 = \frac{1}{2} < 1$$

from which we conclude that indeed $g(x, y) \in C$. It is clear that g is injective and so we are done.

(3) There are at least two possibilities to solve this problem. We present both. The first uses that theorem that a countable union of countable sets is countable. So we present the set X as follows. Let $n \in \mathbb{N}$ ($n = 0$ is included). Define $X_n = \{A \subseteq \mathbb{N} \mid |A| = n\}$. It is clear that

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

so, using the theorem stated, it is enough to show that each X_n is countable (it is obvious that X is infinite). Let n be fixed. We will find an injection $f : X_n \to X$ which then proves X_n is countable. Let $A \in X_n$ and write $A = \{a_1, \cdots, a_n\}$ and moreover assume $a_i < a_{i+1}$ for each $1 \leq i < n$. Then the function $f : X_n \to X$ defined by

$$f(A) = \sum_{i=1}^{n} a_i 10^i$$

is clearly injective and so we are done.

Another possibility for proving this result is by directly constructing an injection $X \to \mathbb{N}$. To achieve that write $\{p_i\}_{i \in \mathbb{N}}$ for the list of all primes numbers with not repetition there are infinitely many primes so this is feasible). Now construct $f : X \to \mathbb{N}$ on $A \in X$ as follows:

$$f(A) = \prod_{i \in A} p_i.$$

The Fundamental Theorem of Arithmetic then guarantees that f is injective and we are done.

Problem C) (<u>new sheet</u>!)

(1) Let $d = gcd(173, 2011)$. Use the Euclidean Algorithm to find d and write d as $x \cdot 2011 + y \cdot 173$ where x and y are integers.

(2) Let a, b be two positive natural numbers and let $gcd(a, b) = c$. Prove that $gcd(a^2, b) \leq c^2$.

Solution

(1) This is completely straightforward following the algorithm in the book (of course this cannot be accepted as an answer in the test).

(2) We use the cirterion of $gcd(u, v)$ which states that it is the smallest positive linear combination of x and y. Since $gcd(a, b) = c$ there exist $x, y \in \mathbb{Z}$ such that $xa + by = c$. Squaring both sides we obtain

$$c^2 = x^2a^2 + b^2y^2 + 2abxy$$

rearranging the right hand sinde yields:

$$x^2a^2 + (bt^2 + 2axy)b$$

and so c^2 is a linear combination of a^2 and b and so, by the criterion above, $gcd(a^2, b) \leq c^2$ as desired.

Problem D) (<u>new sheet!</u>)

(1) Let $(G, *, e)$ be a group. Prove that for every $g, h, k \in G$ holds that $(g * h * k)^{-1} = k^{-1} * h^{-1} * g^{-1}$.

(2) Let $A = \{1, 2, 3\}$. For each element x in the group $Symm(A)$, the group of bijections from A to A, find the least natural number n_x such that in $Symm(A)$ holds $x * x * \cdots * x$ (n_x times) is the unit element.

Solution

(1) To establish the equality it suffices to show that $(g*h*k)*(k^{-1}*h^{-1}g^{-1}) = e$ and that $(k^{-1} * h^{-1}g^{-1}) * (g * h * k) = e$. For the first equality, using the axioms of a group:

$$(g*h*k)*(k^{-1}*h^{-1}g^{-1}) = g*h*k*k^{-1}*h^{-1}*g^{-1} = g*h*h^{-1}*g^{-1} = g*g^{-1} = e.$$

A similar calculation shows the other equality and we are done.

(2) This is a straightforward computation (an answer in the exam will have to include the calculations). The results are that one element has $n_x = 1$, three have $n_x = 2$ and 2 have $n_x = 3$.

Problem E) (<u>new sheet</u>!) For each of the following statements decide if it is true or false. Give a short argument to support your answer.

(1) Let A be a set and $f : A \to A$ a function. If $f \circ f \circ f \circ f \circ f$ is invertible then f is invertible.

(2) There exists a set X and a subset $Y \subseteq X$ such that $X \neq Y$ and $|X| = |Y|$.

(3) For any natural numbers a, b holds that if $a \neq b$ then $gcd(a^b, b^a) = 1$.

(4) Let $(G, *, e)$ be a group. For any two elements $g, h \in G$ holds that $(g*h)^{-1} = g^{-1} * h^{-1}$.

Solution

(1) True. We write $g = f \circ f \circ f \circ f \circ f$. Since g is invertible it is both injective and surjective. We show that this implies f is injective and surjective as well and thus is bijective. Indeed, if f is not injective then $f(a) = f(b)$ for some $a \neq b$. But then clearly $g(a) = g(b)$ which is not possible. So f is injective. Similarly, if f is not surjective then g can't be surjective either (if f misses a value then repeated it 5 times won't be able to hit that element either (a more accurate proof can be given of course)).

(2) True. Let $X = \mathbb{N}$ and $Y = \{2n \mid n \in \mathbb{N}\}$. Clearly $X \neq Y$ and the bijection $f : X \to Y$ given by $f(n) = 2n$ is clearly a bijection so that $|X| = |Y|$.

(3) False. Let $a = 6$ and $b = 2$. Then $a \neq b$ but $gcd(a^b, b^a) = gcd(36, 64) = 4 \neq 1$.

(4) False. Since $(g * h)^{-1} = h^{-1} * g^{-1}$ it follows that if $(g * h)^{-1} = g^{-1} * h^{-1}$ then $h^{-1}g^{-1} = g^{-1}h^{-1}$. Since not every group is commutative this equality does not hold in general and guids us to find a counter example, namely in any non-abelian group. Let $G = Sym(\{1, 2, 3\})$ by the symmetric group. Two non-commuting elements are f and g where $f(1) = 1$, $f(2) = 3$, and $f(3) = 2$ while $g(1) = 2$, $g(2) = 3$, and $g(3) = 1$. A straightforward computation now shows $(f * g)^{-1} \neq f^{-1} * g^{-1}$.