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blok 3, 2013/ 2014
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Analyse in meer variabelen, WISB212

Hertentamen

Family name: ____________ Given name: ____________

Student number: ____________

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use results from the lecture and the book without proving them.

If a map is smooth and given by an "explicit formula" then you may use that it is smooth without proof, unless otherwise stated.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

If you are not able to solve one part of a problem, try to solve the other parts.

You may write in Dutch.

30 points will yield a passing grade 6, and 63 points a grade 10.

Good luck!

1	2	3	4	5	6	7	8	9	10	$\sum$
/5	/7	/5	/9	/4	/7	/4	/6	/12	/7	/66

Problem 1 (derivative of norm, 5pt). Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathbb{R}^n$. Prove that the norm

$$\|\cdot\|:\mathbb{R}^n\setminus\{0\}\to\mathbb{R},\quad \|x\|:=\sqrt{\langle x,x\rangle},$$

is differentiable (on $\mathbb{R}^n\setminus\{0\}$) and calculate its derivative.

Remarks: Here you may **not** use smoothness of a map given by an "explicit formula". However, you may use the fact that every multilinear map is differentiable, and a formula for its derivative.

Problem 2 (nonlinear equation, 7pt). (i) Prove that there exist numbers $a>0$ and $b>0$ with the following properties: For every $y\in(-b,b)$ there exists a unique solution $x=x_y\in\big(1-a,1+a\big)$ of the equation

$$e^{xy}-x=0.$$

Furthermore, the function $y\mapsto x_y$ is smooth.

(ii) Calculate the derivative of this function at 0.

Problem 3 (derivative of inversion map, 5pt). Calculate the derivative of the inversion map

$$\mathrm{GL}(n,\mathbb{R})\ni A\mapsto A^{-1}\in\mathrm{GL}(n,\mathbb{R}).$$

Here $\mathrm{GL}(n,\mathbb{R})$ denotes the set of general linear matrices.

Remark: You may use formulae for the derivatives of a multilinear map and of the map (f,g) in terms of Df and Dg.

Problem 4 (submanifold, 9pt). (i) Draw a picture of the set

$$M:=\big\{x\in\mathbb{R}^3\,\big|\,x_1^4+x_2^4=x_3^4+1\big\}.$$

(ii) Prove that M is a smooth submanifold of $\mathbb{R}^3$ and calculate its dimension.

(iii) Calculate the tangent space to M at any point $x\in M$.

Problem 5 (two-dimensional integral, 4pt). Calculate

$$\int_{-1}^{1}\left(\int_0^1\tan\big(\cos(x_1)x_2\big)\,dx_1\right)dx_2.$$

Problem 6 (volume of ball, 7pt). (i) Prove that the closed unit ball $\overline{B}^3\subseteq\mathbb{R}^3$ is Jordan-measurable.

(ii) Calculate its Jordan-measure (=volume).

Remarks: You may use the fact that for every Jordan-measurable set $S\subseteq\mathbb{R}^m$ and every $c\geq 0$ the set

$$cS=\big\{cx\,\big|\,x\in S\big\}$$

is Jordan-measurable with Jordan-measure

$$|cS| = c^m |S|.$$

You may also use the formulae

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt = \frac{\pi}{2}, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 t \, dt = \frac{4}{3}.$$

Problem 7 (two-dimensional integral, 4pt)**.** Calculate the integral

$$\int_{B_1^2} \big(D_1 X^2 - D_2 X^1\big)(x) \, dx$$

for

$$X(x) := e^{\sin\left(\|x\|^2 - 1\right)} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Here B_1^2 denotes the open unit ball in $\mathbb{R}^2$.

Problem 8 (special linear matrices, 6pt)**.** Consider the set of special linear 2×2-matrices

$$\mathrm{SL}(2, \mathbb{R}) := \big\{ A \in \mathbb{R}^{2\times 2} \,\big|\, \det(A) = 1 \big\}.$$

(i) Prove that $\mathrm{SL}(2,\mathbb{R})$ is a smooth submanifold of $\mathbb{R}^{2\times 2} = \mathbb{R}^4$ and calculate its dimension.

(ii) Compute the tangent space at the identity,

$$T_{\mathbf{1}} \, \mathrm{SL}(2, \mathbb{R}).$$

Problem 9 (maximum, 12pt)**.** Consider the curve

$$M := \big\{ x \in \mathbb{R}^2 \,\big|\, x_1^2 + x_1^{10} + x_2^2 + x_2^{10} = 4 \big\}$$

and the function

$$f : M \to \mathbb{R}, \quad f(x) := x_1 + x_2.$$

(i) Draw a picture of M and several level sets of f.

(ii) Prove that f attains its maximum on M.

(iii) Calculate the maximum of f on M.

Remarks: You may use results from WISB111 (Inleiding Analyse), the fact that a maximum of a function $f \in C^1(M, \mathbb{R})$ is a critical point for f, and injectivity of the functions

$$\mathbb{R} \ni t \mapsto t + 5t^9 \in \mathbb{R}, \quad \mathbb{R} \ni t \mapsto t + t^5 \in \mathbb{R}.$$

Problem 10 (intrinsic boundary, 7pt)**.** Let $d \le n \in \mathbb{N}_0$ and $M \subseteq \mathbb{R}^n$ be a parametrizable C^1-submanifold of dimension d with boundary. (Recall from the lecture that this means that there exists an open C^1-domain $V \subseteq \mathbb{R}^d$ and an injective C^1-immersion $\psi : \overline{V} \to \mathbb{R}^n$ with a continuous inverse and image M.) We define the intrinsic boundary of M to be $\partial M := \psi(\partial V)$ for such a pair (V, ψ).

Show that ∂M is well-defined, i.e., it does not depend on the choice of (V, ψ).