Multidimensional Real Analysis
Tuesday June 26, 2013, 13.30 – 16.30 h.

- Put your name and student number on every sheet that you hand in.
- Do not only give answers, but also prove all statements. When you use a Theorem, show that all conditions are met.
- You are not allowed to use a computer, book or lecture notes.

Good Luck!

1. Let T be the torus in $\mathbb{R}^3$ given by the parametrization

$$\Phi(\alpha,\theta) = ((2+\cos\theta)\cos\alpha, (2+\cos\theta)\sin\alpha, \sin\theta), \qquad -\pi < \alpha, \theta \leq \pi.$$

(a) (10 points) Calculate $\mathrm{Vol}_2(T)$ and show that T is 2-dimensional Jordan measurable.

Let C be the curve on the torus T which is the image for fixed $p \in \mathbb{R}$ of

$$\gamma(t) = ((2+\cos(pt))\cos t, (2+\cos(pt))\sin t, \sin(pt)), \qquad t \in \mathbb{R}.$$

(b) (5 points) Prove that C is a closed curve T if and only if $p \in \mathbb{Q}$ (Hint, investigate the periodicity of γ).

(c) (10 points) Give an integral (simplify as much as is possible) with which you can calculate the length of C (you do not have to solve this integral) and prove that C is 1-dimensional Jordan measurable if and only if $p \in \mathbb{Q}$.

2. Let Ω be the solid ellipsoid in $\mathbb{R}^3$ given by

$$\Omega = \{x \in \mathbb{R}^3 \,|\, \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} < 1\}.$$

(a) (10 points) Calculate $\mathrm{Vol}_3(\Omega)$.

(b) (5 points) Calculate the outer unit normal vector $\nu(x_1,x_2,x_3)$ at $x = (x_1,x_2,x_3) \in \partial\Omega$.

(c) (10 points) In the midterm exam you had to determine the distance from the origin to the geometric tangent plane to $\partial\Omega$ at the point $x = (x_1,x_2,x_3) \in \partial\Omega$. The answer was:

$$\mathrm{d}(0, T_x\partial\Omega) = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{-\frac{1}{2}}.$$

Compute

$$\int_{\partial\Omega} \mathrm{d}(0, T_x\partial\Omega)\, d_2x.$$

Hint: Use e.g. the divergency Theorem of Gauss. Choose a simple vector field such that the formulas nicely match.

3. (a) (10 points) let $f(x) = (x_1, x_2, -2x_3)$ be the vector field in $\mathbb{R}^3$ and let S_- and S_+ be the two hemispheres

$$S_\pm = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1,\ \pm x_3 \geq 0\}.$$

$\nu_\pm$ is the unit normal vector on $S_\pm$ pointed upwarts. Compute both integrals:

$$\int_{S_\pm} \langle f, \nu_\pm \rangle d_2 x$$

and show that they are equal.

We want to generalise this.

Let $H_\pm$ be two hypersurfaces in $\mathbb{R}^n$ parametrized by

$$\Phi_\pm(y_1, y_2, \ldots, y_{n-1}) = (y_1, y_2, \ldots, y_{n-1}, \phi_\pm(y_1, y_2, \ldots, y_{n-1})), \quad y_1^2 + y_2^2 + \cdots y_{n-1}^2 \leq 1.$$

$\phi_\pm$ both C^2, asume that

$$\phi_-(y_1, y_2, \ldots, y_{n-1}) \leq \phi_+(y_1, y_2, \ldots, y_{n-1})$$

and

$$H_+ \cap H_- = \partial H_+ = \partial H_-.$$

$\nu_\pm$ is the unit normal vector on $H_\pm$ with n$^{\text{e}}$-component $(\nu_\pm)_n > 0$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2-vector field with $\operatorname{div} f = 0$.

(b) (10 points) Prove that

$$\int_{H_-} \langle f, \nu_- \rangle (y) d_{n-1} y = \int_{H_+} \langle f, \nu_+ \rangle (y) d_{n-1} y.$$

(c) (10 points) Let $\partial H_\pm$ lie in a hyperplane through the origin, hence

$$\partial H_\pm \subset V_a = \{x \in \mathbb{R}^n \,|\, \langle x, a \rangle = 0\} \quad \text{and let } H_+ \cap V_a = H_- \cap V_a = \partial H_+ = \partial H_-.$$

Note that $a_n \neq 0$, since H_- and H_+ are hypersurfaces. Asume moreover that

$$\langle f(x), a \rangle = 0 \quad \text{for all } x \in V_a.$$

Prove that

$$\int_{H_-} \langle f, \nu_- \rangle (y) d_{n-1} y = \int_{H_+} \langle f, \nu_+ \rangle (y) d_{n-1} y = 0.$$