F. Ziltener

blok 3, 2015/ 2016
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Analyse in meer variabelen, WISB212

Tentamen

Family name: ________________ Given name: ________________

Student number: ________________________________

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Duistermaat and Kolk, without proving it.

Unless otherwise stated, you may also use smoothness of a map that is given by an explicit formula (if the map is indeed smooth).

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

If you are in doubt whether you may use a certain result then please ask!

You may write in Dutch.

30 points will suffice for a passing grade 6.

Good luck!

1	2	3	4	5	6	7	8	$\sum$
/3	/7	/9	/9	/11	/13	/13	/9	/74

Problem 1 (composition with linear map, 3 pt)**.** Let $m, n, p \in \mathbb{N}$, $f : \mathbb{R}^m \to \mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^p$, and $x_0 \in \mathbb{R}^m$. Assume that f is differentiable in x_0 and that T is linear. Prove that $T \circ f$ is differentiable in x_0 and calculate the derivative $D(T \circ f)(x_0)$.

Remark: If you use that a linear map is differentiable then you need to prove this.

Problem 2 (nonlinear equation, 7 pt)**.** (i) Show that there exist numbers $R, r > 0$ such that for every $y \in B^2_r((0,1))$ there exists a unique solution $x = x_y \in B^2_R(0)$ of the equation

$$\begin{pmatrix} 2x_1 + x_2^4 \\ x_1^3 + x_2 + 1 \end{pmatrix} = y.$$

Show that the map $y \mapsto x_y$ is smooth.

(ii) Calculate the derivative of this map at $y = (0, 1)$.

Problem 3 (volume, 9 pt)**.** Consider the solid of revolution

$$S := \left\{ x \in \mathbb{R}^2 \times [0,1] \,\middle|\, x_1^2 + x_2^2 \leq e^{x_3} \right\}.$$

(i) Draw a picture of S.

(ii) Show that S is Jordan measurable.

(iii) Calculate the volume of S.

Remark: In this problem you may use any exercise from the assignments.

Problem 4 (torus is a submanifold, 9 pt)**.** We define

$$\mathbb{T} := \left\{ x \in \mathbb{R}^3 \,\middle|\, \left(\sqrt{x_1^2 + x_2^2} - 2 \right)^2 + x_3^2 = 1 \right\}.$$

(i) Draw a picture of $\mathbb{T}$ and specify 4 points that lie on it and do not lie in the same plane.

(ii) Show that $\mathbb{T}$ is a smooth submanifold of $\mathbb{R}^3$. Calculate its dimension.

(iii) For every $x \in \mathbb{T}$ determine the tangent space $T_x\mathbb{T}$.

(More problems on the back.)

Problem 5 (flux through half-torus, 11 pt)**.** We denote by $\mathbb{T}$ the torus as in Problem 4 and define

$$\Sigma := \mathbb{T} \cap \big(\mathbb{R}^2 \times [0, \infty)\big).$$

This is a smooth submanifold of $\mathbb{R}^3$ with boundary. (You do *not* need to prove this.)

(i) Find a unit normal vector field ν along Σ.

(ii) Calculate $\nu(2, 0, 1)$.

(iii) Calculate the flux

$$\int_\Sigma X \cdot \nu \, dA$$

for the constant vector field $X \equiv e_3$.

Hint: Consider the vector field

$$Y(x) := \frac{1}{2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

Remarks: You do *not need to prove* that in your calculation all signs are correct.

You can solve this part of the problem even if you could not solve part (i).

Problem 6 (maximum, 13 pt)**.** We define

$$\begin{aligned} S &:= \big\{(x, y) \in \mathbb{R}^2 \,\big|\, x^4 - x + y^4 - y = 0\big\}, \\ f &: S \to \mathbb{R}, \quad f(x, y) := x + y. \end{aligned}$$

(i) Prove that f attains its maximum.

(ii) Calculate the maximum of f.

Remark: In part (ii) you may use any exercise from the assignments.

Problem 7 (integral of function, 13 pt)**.** (i) Draw a picture of the set

$$S := \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, 0 < x,\ 1 \le \frac{y}{x} \le 2,\ 2\pi \le xy \le 4\pi \right\}.$$

(ii) We define

$$f : S \to \mathbb{R}, \quad f(x, y) := \frac{y}{x} \sin(xy).$$

Show that f is Riemann-integrable over S. Calculate the integral of f over S.

Hint: Use a theorem from the lecture.

Problem 8 (ball diffeomorphic to cube, 9 pt)**.** Show that for every $n \in \mathbb{N}$ there exists a (surjective) smooth diffeomorphism from the open ball $B^n_1(0)$ to the open cube $(-1, 1)^n$.

Hint: Consider also the set $\mathbb{R}^n$.

Remarks: In this problem you may use any exercise from the assignments.
You will also receive points for solving the problem in the case $n = 2$.