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SOLUTIONS MIDTERM MULTIDIMENSIONAL REAL ANALYSIS

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Exercise 1. ***(30 pt)*** In this exercise, we will compute the total derivative of the *inversion mapping* $G : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ defined by

$$G(x) = \frac{1}{\|x\|^2} x, \tag{1}$$

where $\|x\|$ is the standard norm in $\mathbb{R}^n$, i.e. $\|x\|^2 = \langle x, x \rangle = x^{\mathrm{T}} x$.

(a) (*5 pt*) Describe the action of the mapping (1) geometrically.

G inverts the distance of points to the origin. It preserves all radial rays and interchanges the sphere of radius r centred at the origin with that of radius $\frac{1}{r}$.

(b) (*10 pt*) Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ and $G : U \to \mathbb{R}^n$ be two differentiable mappings. Define $fG : U \to \mathbb{R}^n$ via $(fG)(x) = f(x)G(x),\ x \in U$. Prove that fG is differentiable and

$$D(fG)(x) = f(x)DG(x) + G(x)Df(x), \quad x \in U. \tag{2}$$

This can be done in several ways:

1. Let $x \in U$, then by Hadamard's lemma there exist continuous functions $\phi\colon U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$ and $\Gamma\colon U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$f(y) = f(x) + \phi(y)(y - x) \qquad \text{and} \qquad G(y) = G(x) + \Gamma(y)(y - x)$$

for all $y \in U$. Moreover, $\phi(x) = Df(x)$ and $\Gamma(x) = DG(x)$.

Consequently, we find that

$$\begin{aligned}(fG)(y) &= f(y)\,(G(x) + \Gamma(y)(y - x)) \\ &= f(x)\,G(x) + \phi(y)(y - x)\,G(x) + f(y)\,\Gamma(y)(y - x) \\ &= fG(x) + H(y)(y - x),\end{aligned}$$

where $H\colon U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ is the function given by

$$H(x) = f(x)\,\Gamma(y) + G(x)\,\phi(x).$$

Continuity of H follows by application of the sum and product rules for continuous functions. By applying Hadamard's lemma again, we conclude that fG is differentiable at x, with total derivative

$$D(fG)(x) = H(x) = f(x)\Gamma(x) + G(x)\phi(x) = f(x)DG(x) + G(x)Df(x).$$

2. Because f and G are differentiable by assumption, one can write

$$f(x+h) = f(x) + Df(x)h + R_f(x+h)$$

and

$$G(x+h) = G(x) + DG(x)h + R_G(x+h).$$

Here $R_f \colon U \to \mathbb{R}$ and $R_G \colon U \to \mathbb{R}^n$ satisfy

$$\lim_{h\to 0} \frac{R_f(x+h)}{\|h\|} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \frac{R_G(x+h)}{\|h\|} = 0.$$

By working out the product of these two expressions, one obtains

$$(fG)(x+h) = f(x)G(x) + \big(f(x)DG(x)h + Df(x)h\, G(x)\big) + R_{fG}(x+h),$$

where the final term reads

$$R_{fG}(x+h) = Df(x)h\, DG(x)h + R_f(x+h)G(x) + f(x+h)\, R_G(x+h).$$

Since $h \mapsto G(x)$ and $h \mapsto f(x+h)$ are continuous functions, we obviously have

$$\lim_{h\to 0} \frac{R_f(x+h)}{\|h\|} G(x) = 0 \qquad \text{and} \qquad \lim_{h\to 0} f(x+h) \frac{R_G(x+h)}{\|h\|} = 0.$$

For the first term, we can make the estimate

$$\frac{|Df(x)h|\, \|DG(x)h\|}{\|h\|} \leq \frac{\|Df(x)\|\, \|DG(x)\|\, \|h\|^2}{\|h\|} = \|Df(x)\|\, \|DG(x)\|\, \|h\|,$$

so this also vanishes in the limit for $h \to 0$. We conclude that

$$\lim_{h\to 0} \frac{R_{fG}(x+h)}{\|h\|} = 0.$$

Hence, fG is differentiable and its total derivative is given by

$$\begin{aligned} D(fG)(x)h &= f(x)DG(x)h + Df(x)h\, G(x) \\ &= \big(f(x)DG(x) + G(x)Df(x)\big)h. \end{aligned}$$

3. One can use the fact that an $\mathbb{R}^n$-valued function is differentiable if and only if all of its components are.

   For $1 \leq i \leq n$, the i-th component of fG is given by $(fG)_i(x) = f(x)G_i(x)$ and is a product of scalar functions. Both f and G_i are differentiable by assumption, so one may conclude from the product rule that their product is as well, with total derivative

$$D(fG)_i(x) = G_i(x)\, Df(x) + f(x)\, DG_i(x).$$

   Since each of its components are differentiable, the original function fG is as well and its derivative is given by

$$D(fG)(x)h = \begin{pmatrix} D(fG)_1(x)h \\ \vdots \\ D(fG)_n(x)h \end{pmatrix} = \begin{pmatrix} G_1(x)\, Df(x)h + f(x)\, DG_1(x)h \\ \vdots \\ G_n(x)\, Df(x)h + f(x)\, DG_n(x)h. \end{pmatrix}$$

   More concisely, we read off that $D(fG)(x) = G(x)Df(x) + f(x)\, DG(x)$.

(c) (*5 pt*) Using (2) with $f(x) = \|x\|^2$, compute the total derivative $DG(x)$ of the mapping (1) for $x \in U$, where $U = \mathbb{R}^n \setminus \{0\}$.

In our specific case, we have that $f(x)\,G(x) = x$ for all $x \in \mathbb{R}^n \setminus \{0\}$, so $fG = \mathrm{id}$. From this, it follows that

$$D(fG) = G\,Df + f\,DG = D\mathrm{id} = \mathrm{id}.$$

We know the derivative of $f\colon x \mapsto \|x\|^2$ to be $Df(x)h = 2\,\langle x, h\rangle = 2\,x^{\mathrm{T}}h$, so the above identity tells us that

$$\begin{aligned} DG(x) &= f(x)^{-1}(\mathrm{id} - G(x) \cdot Df(x)) \\ &= \frac{1}{\|x\|^2}\left(\mathrm{id} - \frac{x}{\|x\|^2} \cdot 2\,x^{\mathrm{T}}\right) = \frac{1}{\|x\|^2}\,A(x), \end{aligned}$$

where for $x \in \mathbb{R} \setminus \{0\}$, $A(x)$ denotes the matrix

$$A(x) = I - 2\,\frac{x\,x^{\mathrm{T}}}{\|x\|^2}.$$

(d) (*10 pt*) Show that for $x \in U$ holds $DG(x) = \|x\|^{-2}A(x)$, where $A(x)$ is represented by an orthogonal matrix, i.e. $A^{\mathrm{T}}(x)A(x) = I$.

We recognise $A(x)$ as the matrix representing a reflection in the plane perpendicular to x. We will verify that this is an orthogonal transformation.

Because $A^{\mathrm{T}}(x) = A(x)$, we see that

$$A^{\mathrm{T}}(x)A(x) = \left(I - 2\,\frac{x\,x^{\mathrm{T}}}{\|x\|^2}\right)^2 = I^2 - 4\,\frac{x\,x^{\mathrm{T}}}{\|x\|^2} + 4\,\frac{x\,x^{\mathrm{T}}x\,x^{\mathrm{T}}}{\|x\|^4}.$$

Because $x^{\mathrm{T}}x = \|x\|^2$, the last two terms cancel out and we may conclude that $A^{\mathrm{T}}(x)A(x) = I^2 = I$.

Exercise 2 (*30 pt*). Let $a, b, c > 0$ and let M be the *ellipsoid* in $\mathbb{R}^3$ defined as

$$M = \left\{x \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1\right\}.$$

(a) (*10 pt*) Find the tangent space of M at $x \in M$.

Introduce $g : \mathbb{R}^3 \to \mathbb{R}$ by

$$g(x) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2},$$

so that $M = \{x \in \mathbb{R}^3 : g(x) = 1\}$. A simple computation shows that the derivative of g at $x \in \mathbb{R}^3$ reads

$$Dg(x) = \begin{pmatrix} \frac{2\,x_1}{a^2} & \frac{2\,x_2}{b^2} & \frac{2\,x_3}{c^2} \end{pmatrix}, \tag{3}$$

which is non-zero for all $x \neq 0$. Hence g is a submersion at every point $x \in M$ and its geometric tangent space at x is given by

$$\tilde{T}_x M = \{y \in \mathbb{R}^3 \mid Dg(x)(y - x) = 0\} = \{y \in \mathbb{R}^3 \mid Dg(x)y = 2\}.$$

For this have used that $Dg(x)x = \dfrac{2\,x_1}{a^2}x_1 + \dfrac{2\,x_2}{b^2}x_2 + \dfrac{2\,x_3}{c^2}x_3 = 2\,g(x) = 2.$

(b) (*20 pt*) Compute the distance from the origin to the geometric tangent plane to M at an arbitrary point $x \in M$.

The distance from the origin to the tangent plane at $x \in M$ can be found through either a geometric argument or by applying the method of Lagrange multipliers.

1. The distance from the origin to the plane will be equal to the length of the component of $x \in \tilde{T}_x M$ orthogonal to it. Since we know that $\operatorname{grad} g(x) = [Dg(x)]^{\mathrm{T}}$ is orthogonal to the tangent space $T_x M$, this length will be given by

$$d(0, \tilde{T}_x M) = \frac{\langle x, \operatorname{grad} g(x) \rangle}{\| \operatorname{grad} g(x) \|} = \frac{Dg(x)x}{\|Dg(x)\|}.$$

   We have already computed the numerator $Dg(x)x = 2$, and the denominator can be read off from equation (3). We thus obtain

$$d(0, \tilde{T}_x M) = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4} \right)^{-\frac{1}{2}}.$$

2. One may also arrive at this answer through the method of Lagrange multipliers. The distance $d(0, \tilde{T}_x M)$ is then obtained by minimising the function $f\colon x \mapsto \|x\|^2$ on the geometric tangent plane $\tilde{T}_x M$. Since the plane $\tilde{T}_x M \subseteq$ is a closed subset of $\mathbb{R}^3$, f assumes a minimum on it at some point $y_0 \in \tilde{T}_x M$ and the distance from the origin to the plane will be the square root of this minimum. (NB: The intersection $\tilde{T}_x M \cap \overline{B(0,R)}$ is compact and non-empty for an appropriately chosen $R > 0$. The norm assumes a minimum on it, which is in fact a global minimum.)

   The point $y_0 \in \tilde{T}_x M$ will necessarily be a critical point for f, which means that $\operatorname{grad} f(y_0) = 2\,y_0$ is orthogonal to $\tilde{T}_x M$, hence parallel to $\operatorname{grad} g(x)$. Let $\lambda \in \mathbb{R}$ be such that $y_0 = \lambda \operatorname{grad} g(x)$, then we see that (since $y_0 \in \tilde{T}_x M$)

$$Dg(x)y_0 = \langle \operatorname{grad} g(x), \lambda \operatorname{grad} g(x) \rangle = \lambda \, \| \operatorname{grad} g(x) \|^2 = 2.$$

   We derive that $\lambda = 2 \, \| \operatorname{grad} g(x) \|^{-2}$ and that therefore

$$\|y_0\| = |\lambda| \, \| \operatorname{grad} g(x) \| = \frac{2}{\| \operatorname{grad} g(x) \|} = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4} \right)^{-\frac{1}{2}}.$$

   This confirms our earlier conclusion.

3. The critical point described in part 2 also corresponds to a critical point for the Lagrange function

$$L\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \qquad (y, \lambda) \mapsto f(y) - \lambda\, h(y),$$

   where $f(y) = \|y\|^2$ and $h(y) = Dg(x)y - 2$.

   Since $Df(y) = 2\,y^{\mathrm{T}}$ and $Dh(y) = Dg(x)$, the equation $DL(y, \lambda) = 0$ becomes

$$DL(y) = (Df(y) - \lambda\, Dh(y), h(y)) = (2\,y^{\mathrm{T}} - \lambda\, Dg(x), Dg(x)y - 2) = 0.$$

   Solving this system of equations essentially comes down to following the steps from option 2.

Exercise 3. ***(40 pt)*** Here, we will study a representation of the *Möbius Strip* in $\mathbb{R}^3$.

(a) *(5 pt)* Let $D = \{(\theta, t) \in \mathbb{R}^2 : -\pi < \theta < \pi, -1 < t < 1\}$ and let $\Phi : D \to \mathbb{R}^3$ be defined by

$$\Phi(\theta, t) = \begin{pmatrix} \left(2 + t\cos\left(\frac{\theta}{2}\right)\right)\cos\theta \\ \left(2 + t\cos\left(\frac{\theta}{2}\right)\right)\sin\theta \\ t\sin\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

Prove that Φ is an immersion at any point in D.

The function Φ is clearly C^∞, and we can explicitly compute its derivative

$$\begin{aligned} D\Phi(\theta, t) &= (\partial_\theta \Phi(\theta, t) \quad \partial_t \Phi(\theta, t)) \\ &= \begin{pmatrix} -\frac{1}{2}t\sin(\frac{1}{2}\theta)\cos\theta - (2 + t\cos(\frac{1}{2}\theta))\sin\theta & \cos(\frac{1}{2}\theta)\cos\theta \\ -\frac{1}{2}t\sin(\frac{1}{2}\theta)\sin\theta + (2 + t\cos(\frac{1}{2}\theta))\cos\theta & \cos(\frac{1}{2}\theta)\sin\theta \\ \frac{1}{2}t\cos(\frac{1}{2}\theta) & \sin(\frac{1}{2}\theta) \end{pmatrix} \end{aligned}$$

There are at least three ways to verify that $D\Phi(\theta, t)$ is injective for all $(\theta, t) \in D$, so that Φ is an immersion.

1. One can compute the determinant of the upper 2×2 block of $D\Phi(\theta, t)$. This determinant equals
$$-(2 + t\cos(\tfrac{1}{2}\theta))\cos(\tfrac{1}{2}\theta).$$
This is non-zero for all $(\theta, t) \in D$, meaning that $D\Phi(\theta, t)$ has rank 2 and that Φ is an immersion.

2. One can also decompose
$$D\Phi(\theta, t) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}t\sin(\frac{1}{2}\theta) & \cos(\frac{1}{2}\theta) \\ 2 + t\cos(\frac{1}{2}\theta) & 0 \\ \frac{1}{2}t\cos(\frac{1}{2}\theta) & \sin(\frac{1}{2}\theta) \end{pmatrix}.$$
Since $2 + t\cos(\frac{1}{2}\theta) > 0$ for $(\theta, t) \in D$, the two columns of the 3×2-matrix on the second line are linearly independent. Because the square matrix that was factored out is invertible, we conclude that $D\Phi(\theta, t)$ is injective and that Φ is therefore an immersion.

3. Another option is calculating the cross product $\partial_\theta \Phi(\theta, t) \times \partial_t \Phi(\theta, t)$. The third component of this cross product is
$$-(2 + t\cos(\tfrac{1}{2}\theta))\cos(\tfrac{1}{2}\theta)(\sin^2\theta + \cos^2\theta) = -(2 + t\cos(\tfrac{1}{2}\theta))\cos(\tfrac{1}{2}\theta).$$
This is non-zero for all $(\theta, t) \in D$, which means that the columns of $D\Phi(\theta, t)$ are linearly independent. We conclude that $D\Phi(\theta, t)$ has rank 2 and that Φ is an immersion.

(b) *(10 pt)* Show that $\Phi : D \to \Phi(D)$ is invertible and that the inverse mapping is continuous. Use this to conclude that $V = \Phi(D)$ is a C^∞ submanifold in $\mathbb{R}^3$ of dimension 2.

For $(x, y) \in \mathbb{R}^2$ of the form $(x, y) = \rho(\cos\phi, \sin\phi)$ with $\rho > 0$ and $\phi \in]-\pi, \pi[$, one can recover $\rho = \sqrt{x^2 + y^2}$ and $\phi = 2\arctan(\frac{y}{\rho + x})$. We therefore define

$$\rho \colon \mathbb{R}^2 \setminus \{(0,0)\} \to]0, \infty[, \qquad \text{and} \qquad \phi \colon \mathbb{R}^2 \setminus \{(x, 0) \mid x \le 0\} \to]-\pi, \pi[$$

by setting

$$\rho(x,y) = \sqrt{x^2+y^2} \qquad \text{and} \qquad \phi(x,y) = 2\arctan\left(\frac{y}{\rho(x,y)+x}\right).$$

Since all functions involved are smooth on their domain, ρ and ϕ are C^∞ as well.

If $(x,y,z) = \Phi(\theta,t)$, then we see that $\theta = \phi(x,y)$ and $2 + t\cos(\frac{1}{2}\theta) = \rho(x,y)$, from which t can also be obtained since $\cos(\frac{1}{2}\theta) \neq 0$. This leads us to conclude that the map $\Psi\colon \mathbb{R}^3 \setminus \{(x,0,z) \in \mathbb{R}^3 \mid x \leq 0\} \to]-\pi,\pi[\times\mathbb{R}$ such that

$$\Psi(x,y,z) = \begin{pmatrix} \phi(x,y) \\ \frac{\rho(x,y)-2}{\cos(\frac{1}{2}\phi(x,y))} \end{pmatrix}$$

is a left-inverse of Φ, i.e. $\Psi \circ \Phi = \mathrm{id}\colon D \to D$. We deduce that Φ is injective and that its inverse is the restriction $\Psi|_{\Phi(D)}\colon \Phi(D) \to D$.

Since we have described it as a composition of continuous functions, Ψ is also continuous, as is the restriction $\Psi|_{\Phi(D)}\colon \Phi(D) \to D$. We conclude that Φ is a C^∞ embedding and that its image $\Phi(D)$ is therefore a 2-dimensional C^∞ submanifold of $\mathbb{R}^3$.

(c) (*5 pt*) Prove that any point $x \in V$ satisfies $g(x) = 0$, where $g : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$g(x) = 4x_2 + 4x_1x_3 - x_2(x_1^2+x_2^2+x_3^2) + 2x_3(x_1^2+x_2^2). \tag{4}$$

Notice that each term in g has factor $(2 + t\cos(\frac{1}{2}\theta))$. This implies

$$\begin{aligned} g &= \left(2+t\cos(\tfrac{1}{2}\theta)\right)\left[4\sin\theta + 4t\cos\theta\sin(\tfrac{1}{2}\theta) - \sin\theta\left(4+4t\cos(\tfrac{1}{2}\theta)+t^2\right)\right. \\ &\qquad \left. +\ 2t\sin(\tfrac{1}{2}\theta)\left(2+t\cos(\tfrac{1}{2}\theta)\right)\right] \\ &= \left(2+t\cos(\tfrac{1}{2}\theta)\right)\left[4t\left(\cos\theta\sin(\tfrac{1}{2}\theta) - \sin\theta\cos(\tfrac{1}{2}\theta)\right)\right. \\ &\qquad \left. -\ t^2\sin\theta + 4t\sin(\tfrac{1}{2}\theta) + 2t^2\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)\right] \\ &= \left(2+t\cos(\tfrac{1}{2}\theta)\right)\left[-4t\sin(\tfrac{1}{2}\theta) - t^2\sin\theta + 4t\sin(\tfrac{1}{2}\theta) + t^2\sin\theta\right] \ = \ 0, \end{aligned}$$

since

$$2\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta) = \sin\theta$$

and

$$\cos\theta\sin(\tfrac{1}{2}\theta) - \sin\theta\cos(\tfrac{1}{2}\theta) = \sin(\tfrac{1}{2}\theta - \theta) = -\sin(\tfrac{1}{2}\theta).$$

We conclude that $g(\Phi(\theta,t)) = 0$ for all $(\theta,t) \in D$.

(d) (*10 pt*) The Möbius strip is the closure $M = \overline{V}$ of V in $\mathbb{R}^3$. Verify that the circle $S = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 4 \text{ and } x_3 = 0\}$ belongs to M. Give a parametrization of S by $\theta \in]-\pi,\pi]$. Prove that g introduced by (4) is a submersion at any point $x \in S$ except for $x = (-2,0,0)$.

One can parametrise the circle S by $f\colon]-\pi,\pi] \to \mathbb{R}^3, \theta \mapsto (2\cos\theta, 2\sin\theta, 0)$. Note that $f(]-\pi,\pi[) \subseteq \Phi(D)$ because $f(\theta) = \Phi(\theta,0)$ for $\theta \in]-\pi,\pi[$.

The fact that f is continuous then tells us that

$$f(]-\pi,\pi]) = f\left(\overline{]-\pi,\pi[}\right) \subseteq \overline{f(]-\pi,\pi[)} \subseteq \overline{V} = M,$$

where $\overline{]-\pi,\pi[} =]-\pi,\pi]$ denotes the closure of $]-\pi,\pi[$ in $]-\pi,\pi]$.

One way to derive this is by writing $\pi = \lim_{n\to\infty} a_n$ for some sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in]-\pi,\pi[$, so that $f(\pi) = \lim_{n\to\infty 0} f(a_n)$ by the continuity of f. From this we conclude that $f(\pi)$ is a limit point of $f(]-\pi,\pi[) \subseteq V$ and is therefore in the closure $M = \overline{V}$.

The gradient of g can easily be computed, and reads

$$\operatorname{grad} g(x) = \begin{pmatrix} 4x_3 - 2\,x_1\,x_2 + 4\,x_1\,x_3 \\ 4 - (x_1^2 + x_2^2 + x_3^2) - 2\,x_2^2 + 4\,x_3\,x_2 \\ 4\,x_1 - 2\,x_2\,x_3 + 2\,(x_1^2 + x_2^2) \end{pmatrix}$$

By plugging in $x = f(\theta)$, we obtain the expression

$$\operatorname{grad} g(f(\theta) = \begin{pmatrix} -8\cos\theta\sin\theta \\ 4 - 4 - 8\sin^2\theta \\ 8\cos\theta + 8 \end{pmatrix} = 4 \begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\theta) - 1 \\ 2\,(\cos\theta + 1) \end{pmatrix}.$$

The last component is non-zero for all $\theta \in]\pi,\pi[$, while for $\theta = \pi$ all components vanish. Thus, g is a submersion at every point of S except for $f(\pi) = (-2,0,0)$.

This shows that V is a submanifold at every point in $S \cap V$, corroborating the conclusion from part (b).

(e) *(10 pt)* Show that $n_0 = (0,0,1) \in \mathbb{R}^3$ is orthogonal to the tangent space $T_{\Phi(0,0)}V$. Compute a continuous vector-valued function $n :]-\pi,\pi[\to \mathbb{R}^3$ such that $n(0) = n_0$ and for all $-\pi < \theta < \pi$ the vector $n(\theta) \in \mathbb{R}^3$ is orthogonal to $T_{\Phi(\theta,0)}V$ while $\|n(\theta)\| = 1$. Verify that

$$\lim_{\theta\to\pi} n(\theta) = -\lim_{\theta\to-\pi} n(\theta).$$

Here again several approaches are possible.

1. Since we have shown that the function g is a submersion at $x = \Phi(\theta,0) = f(\theta)$ for $\theta \in]-\pi,\pi[$ and $V \subseteq g^{-1}(\{0\})$, we also know that the gradient $\operatorname{grad} g(x)$ is normal to the tangent space $T_{\Phi(\theta,0)}V$. Because $\operatorname{grad} g(f(0)) = (0,0,16)$, it follows that also $n_0 = (0,0,1)$ is orthogonal to $T_{\Phi(0,0)}V$.

   The function n described in the exercise is obtained by normalising the vectors $\operatorname{grad} g(f(\theta))$ for $\theta \in]-\pi,\pi[$ and setting

   $$n(\theta) = \frac{\operatorname{grad} g(f(\theta))}{\|\operatorname{grad} g(f(\theta))\|} = \frac{1}{4\,|\cos(\frac{1}{2}\theta)|} \begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\,\theta) - 1 \\ 2\,(\cos\theta + 1) \end{pmatrix}.$$

   A few trigonometric identities have been applied to obtain the final, simplified expression:

   $$\begin{aligned} &\sin^2(2\,\theta) + (\cos(2\,\theta) - 1)^2 + 4\,(\cos\theta + 1)^2 \\ &\quad = \sin^2(2\theta) + \cos^2(2\theta) - 2\cos(2\theta) + 1 + 4\cos^2\theta + 8\cos\theta + 4 \\ &\quad = 6 - 2(\cos^2\theta - \sin^2\theta) + 4\cos^2\theta + 8\cos\theta \\ &\quad = 8 + 8\cos\theta = 16\cos^2(\tfrac{1}{2}\theta). \end{aligned}$$

   We note that $|\cos(\frac{1}{2}\theta)| = \cos(\frac{1}{2}\theta)$ for $-\pi \le \theta \le \pi$, so that the limits $\lim_{\theta\to\pm\pi} n(\theta)$ can be obtained by applying l'Hôpital's rule:

$$\begin{aligned}
\lim_{\theta\to\pm\pi} n(\theta) &= \lim_{\theta\to\pm\pi} \frac{1}{4\cos(\frac{1}{2}\theta)} \begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\,\theta)-1 \\ 2\,(\cos\theta+1) \end{pmatrix} \\
&= \lim_{\theta\to\pm\pi} \frac{1}{\frac{\mathrm{d}}{\mathrm{d}\theta}4\cos(\frac{1}{2}\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta} \begin{pmatrix} -\sin(2\,\theta) \\ \cos(2\,\theta)-1 \\ 2\,(\cos\theta+1) \end{pmatrix} \\
&= \lim_{\theta\to\pm\pi} \frac{1}{-2\sin(\frac{1}{2}\theta)} \begin{pmatrix} -2\cos(2\,\theta) \\ -2\sin(2\,\theta) \\ -2\,\sin\theta \end{pmatrix}.
\end{aligned}$$

This is just the limit of a continuous function, so we read off that

$$\lim_{\theta\to\pi} n(\theta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lim_{\theta\to-\pi} n(\theta) = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2. A somewhat different approach involves the cross product $\partial_\theta\Phi(\theta,t)\times\partial_t\Phi(\theta,t)$ of the partial derivatives of part (a). Because Φ is an immersion, this cross-product is non-vanishing for every $(\theta,t)\in D$, and is orthogonal to the tangent space $T_{\Phi(\theta,t)}$.

   Since at $\partial_\theta\Phi(0,0) = (0,2,0)$ and $\partial_t\Phi(0,0) = (1,0,0)$, we have $\partial_\theta\Phi(0,0)\times\partial_t\Phi(0,0) = (0,0,-2)$ and we can again conclude that $n_0 = (0,0,1)$ is orthogonal to $T_{\Phi(0,0)}V$.

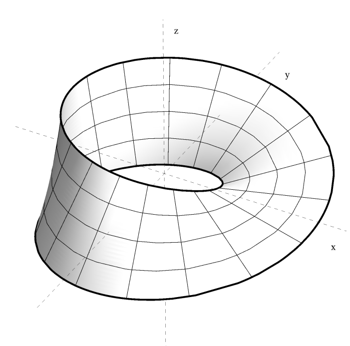
   Because $\partial_\theta\Phi(0,0)\times\partial_t\Phi(0,0)$ and n_0 are pointing in opposite directions, an additional minus sign needs to be introduced in the definition of n, so that

$$n(\theta) = \frac{-\partial_\theta\Phi(\theta,0)\times\partial_t\Phi(\theta,0)}{\|\partial_\theta\Phi\theta,0)\times\partial_t\Phi(\theta,0)\|}.$$

   This will lead to the same answer.

(f) (Bonus: *5 pt*) Sketch the set M and describe its geometry.

The Möbius strip M is a smooth 2-dimensional connected manifold with boundary in $\mathbb{R}^3$. It is similar to a cylinder in the sense that it can be described as the union of a continuous family of line segments over the circle, but these line segments are gradually twisted as one goes around the circle. This happens in such a way that if one follows a line segment around the circle once, its end points are interchanged. (It is a non-trivial fibre bundle.)



The Möbius strip is non-orientable, which can be expressed by saying that it has only ‘one side’. This was demonstrated in part (e), where a vector normal to the surface was continuously transported around the loop once and ended up on the ‘other side’.