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blok 4, 2016/ 2017
27 juni, 2017

Analyse in meer variabelen, WISB212

Tentamen

Family name: ____________________ Given name: ____________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You may write in Dutch.

30 points will suffice for a passing grade 6.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

(Instructions continued on reverse.)

1	2	3	4	5	6	7	8	$\sum$
/5	/8	/10	/6	/10	/13	/8	/9	/69

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Duistermaat and Kolk, without proving it.

Unless otherwise stated, you may use the following without proof:

- Smoothness of a map that is given by *one* explicit formula (not different formulae for different cases).
- We define

$$U := \mathbb{R}^2 \setminus \big((-\infty, 0] \times \{0\}\big), \quad V := (0, \infty) \times (-\pi, \pi),$$
$$\Psi : V \to U,\ \Psi(r, \varphi) := \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}.$$

  The map Ψ is a C^∞-diffeomorphism.

- A formula for $\det(D\Psi)$ with Ψ as above.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

If you are in doubt whether you may use a certain result then please ask!

Good luck!

Problem 1 (first and second derivative, 5 pt)**.** Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) := x_1 x_2.$$

Explain what sort of objects the derivative Df and the second derivative $D(Df)$ are (as defined in the lecture or in the book by Duistermaat and Kolk). Calculate Df and $D(Df)$.

Problem 2 (spherical coordinates, 8 pt)**.** We define

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(r, \varphi, \theta) := r\big(\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta\big).$$

(i) Draw the images under f of the planes $\{1\} \times \mathbb{R}^2$, $\mathbb{R} \times \{\pi/2\} \times \mathbb{R}$, and $\mathbb{R}^2 \times \{\pi/4\}$.

(ii) Prove that every point in $(0, \infty) \times \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ admits an open neighbourhood V such that the restriction $f : V \to f(V)$ is a smooth diffeomorphism.

Remark: If you use the determinant of a certain matrix then you need to calculate it.

Problem 3 (tangent spaces to logarithmic spiral, 10 pt)**.** (i) Draw a picture of the logarithmic spiral

$$M := \big\{e^y\big(\cos y, \sin y\big) \,\big|\, y \in \mathbb{R}\big\}.$$

(ii) Prove that this is a smooth submanifold of $\mathbb{R}^2$. Calculate its dimension.

(iii) Calculate the tangent space of M at any point.

(iv) Determine $T_{(1,0)}M$.

Problem 4 (iterated integral, 6 pt)**.** Prove that the iterated integral

$$\int_0^1 \int_y^1 \sin\left(\frac{2\pi y}{x}\right)\, dx dy$$

is well-defined and calculate it.

Problem 5 (integral over piece of cake, 10 pt)**.** Let $\varphi_0 \in (0, \pi)$. We define

$$S := \big\{r\big(\cos\varphi, \sin\varphi\big) \,\big|\, r \in [0, 1],\ \varphi \in [0, \varphi_0]\big\},$$
$$f : S \to \mathbb{R}, \quad f(x) := e^{\|x\|^2}.$$

(i) Show that f is properly Riemann integrable (over S).

(ii) Calculate the integral $\int_S f(x)\, dx$.

(More problems on the back.)

Problem 6 (line integral, 13 pt)**.** Consider the set

$$C := \{(x,y) \in \mathbb{R}^2 \,|\, x^6 - x + y^6 - y = 0\}.$$

(i) Prove that C is compact.

(ii) Prove that C is a smooth submanifold of $\mathbb{R}^2$. Calculate its dimension.

(iii) Find a unit tangent vector field T along C.

(iv) Calculate the line integral

$$\int_C X \cdot T \, ds$$

of the vector field

$$X(x,y) := \begin{pmatrix} e^{x^2} \\ \cos(y^3) \end{pmatrix}.$$

Hint: Use a theorem from the lecture.

Problem 7 (derivative of inversion map, 8 pt)**.** (i) Prove that the inversion map

$$\mathrm{GL}(n,\mathbb{R}) \ni A \mapsto A^{-1} \in \mathrm{GL}(n,\mathbb{R}) \tag{1}$$

is smooth. (Here we equip $\mathbb{R}^{n\times n}$ with an arbitrary norm.)

(ii) Calculate the derivative of the map (1).

Remark: In this problem you may **not** use smoothness of a map that is given by an explicit formula. However, you may use that every bilinear map is smooth, as well as a formula for its derivative.

Hint: Use a theorem from the lecture.

Problem 8 (interchanging the order of integration, 9 pt)**.** Find a function $f : (0,1)\times(0,1) \to \mathbb{R}$, such that the following holds:

- The functions $f(\cdot, y), f(x, \cdot) : (0,1) \to \mathbb{R}$ are (properly) Riemann integrable for all $x, y \in (0,1)$.
- The function $(0,1) \ni y \mapsto \int_0^1 f(x,y)\,dx \in \mathbb{R}$ is Riemann integrable with integral

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = 0.$$

- For every $a \in (0,1)$ the function $(a,1) \ni x \mapsto \int_0^1 f(x,y)\,dy$ is Riemann integrable, and

$$\int_a^1 \left(\int_0^1 f(x,y)\,dy \right) dx \to \infty, \quad \text{as } a \to 0. \tag{2}$$

Remark: Condition (2) means that for every $C \in \mathbb{R}$ there exists $a_0 \in (0,1)$, such that $\int_a^1 \ldots \geq C$ for every $a \in (0, a_0]$.

Hint: It may help to subdivide $(0,1) \times (0,1)$ into pieces and define f on each piece separately.