Exam Analyse in Meer Variabelen 2018-06-26, 9:00–12:00

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Justify your anwers with complete arguments, unless specified otherwise. If you use results from the books or lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 40. The final grade will be obtained from your total score through division by 4.
- You are free to write the solutions either in English, or in Dutch.

Good Luck !

10 pt total **Exercise 1.** Put $U = (0,\infty) \times (0, 2\pi)$ and define $\Phi : U \to \mathbb{R}^2$ by $\Phi(r,\varphi) = (r\cos\varphi, r\sin\varphi)$.

3 pt (a) Calculate $D\Phi(r,\varphi)$ and show that Φ is a C^∞ diffeomorphism onto an open subset V of $\mathbb{R}^2$.

4 pt (b) For $f : V \to \mathbb{R}$ a C^1-function. Show that for all $(r,\varphi) \in U$ we have

$$([D_1 f] \circ \Phi)(r,\varphi) = \left[\cos\varphi \frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial}{\partial \varphi}\right](f\circ\Phi)(r,\varphi)$$
$$([D_2 f] \circ \Phi)(r,\varphi) = \left[\sin\varphi \frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi \frac{\partial}{\partial \varphi}\right](f\circ\Phi)(r,\varphi)$$

Hint: first calculate $D(f\circ\Phi)(r,\varphi)$.

3 pt (c) If f is C^2 and $\varphi \mapsto f(\Phi(r,\varphi))$ is constant for every $r > 0$, show that

$$(\Delta f)\circ\Phi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right](f\circ\Phi) \qquad \text{on } U.$$

10 pt total **Exercise 2.** We assume that M and N are C^k submanifolds of $\mathbb{R}^n$ of dimensions p and q, with $p+q=n$. Here $1 \le k \le \infty$. Furthermore, we assume that for every $x \in M\cap N$ we have $T_xM \oplus T_xN = \mathbb{R}^n$.

2 pt (a) Let $x^0 \in M \cap N$. Show that there exist an open neighborhood $U \ni x^0$ in $\mathbb{R}^n$ and C^k submersions $f : U \to \mathbb{R}^q$ and $g : U \to \mathbb{R}^p$ such that $f(x^0) = g(x^0) = 0$ and

$$f^{-1}(\{0\}) = M\cap U, \quad \text{and} \quad g^{-1}(\{0\}) = N \cap U.$$

Note that f and g are required to have the same domain U.

2 pt (b) Show that for every point $x \in U$ the map $F := (g,f) : U \to \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$ has differential $DF(x) \in \mathrm{Aut}(\mathbb{R}^n)$.

2 pt (c) Show that there exists an open neigbhorhood V of 0 in $\mathbb{R}^n$ and a diffeomorphism Φ from V onto an open neighborhood of x^0 in $\mathbb{R}^n$ such that

$$\Phi(V \cap (\mathbb{R}^p \times \{0\})) = M \cap \Phi(V) \quad \text{and} \quad \Phi(V \cap (\{0\} \times \mathbb{R}^q)) = N \cap \Phi(V).$$

2 pt (d) Show that there exists an open neighborhood $\mathcal{O}$ of x^0 in $\mathbb{R}^n$ such that $\mathcal{O} \cap (M \cap N) = \{x^0\}$.

2 pt (e) If in addition M and N are known to be compact, show that $M \cap N$ is finite.

10 pt total **Exercise 3.** We assume that $B = I_1 \times I_2$ is a rectangle in $\mathbb{R}^2$, where $I_j = [a_j, b_j]$, for $j = 1,2$. Let $f = 1_S$, where $S \subset B$.

2 pt (a) If $S \subset \partial B$, show that f is Riemann integrable with zero integral. Hint: compare f with $1_{\partial B}$.

From now on we assume that $\mathrm{int}(B) \subset S \subset B$.

2 pt (b) Show that f is Riemann integrable while $\int_{\mathbb{R}^2} f(x)\, dx = \mathrm{vol}(B)$.

2 pt (c) Show that for all $a_1 < u < b_1$ we have

$$\overline{\int_{I_2}} f(u,v)\, dv = \underline{\int_{I_2}} f(u,v)\, dv.$$

2 pt (d) Argue that the equality in (c) need not hold for $u = b_1$.

2 pt (e) Prove that

$$\int_{I_1} \overline{\int_{I_2}} f(u,v)\, dv\, du = \int_{I_1} \underline{\int_{I_2}} f(u,v)\, dv\, du = \int_{\mathbb{R}^2} f(x)\, dx.$$

10 pt total **Exercise 4.** For $n \geq 1$ we consider the set $K_n = \{x \in \mathbb{R}^2 \mid \frac{1}{n} \leq \|x\| \leq 1\}$. Furthermore, we define two sets $K_n^{\pm} := \{x \in K_n \mid \pm x_1 \geq 0\}$.

2 pt (a) Show that the sets K_n^+ and K_n^- are compact and Jordan measurable.

5 pt (b) Give a detailed proof that the function $f : x \mapsto \|x\|^{-1}$ is Riemann integrable over K_n^+ and over K_n^- and that

$$\int_{K_n^{\pm}} \|x\|^{-1}\, dx = \pi(1 - \frac{1}{n})$$

3 pt (c) Prove that f is absolutely Riemann integrable over the punctured open unit disk $D \setminus \{0\} = \{x \in \mathbb{R}^2 \mid 0 < \|x\| < 1\}$ and compute

$$\int_{D \setminus \{0\}} \|x\|^{-1}\, dx.$$