Exam Analyse in Meer Variabelen 2018-06-26, 9:00–13:00

Solution 1

(a) Clearly, Φ is C^∞ and injective. By a simple calculation we have

$$D\Phi(r,gf) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

It now follows that $\det \Phi(r,\varphi) = r \neq 0$ for all all $(r,\varphi) \in U$. By the inverse function theorem it follows that Φ is a diffeomorphism from U onto an open subset V of $\mathbb{R}^2$.

(b) We put $f^* = f \circ \Phi$. By the chain rule it follows that for $(r,\varphi) \in U$ we have

$$D(f^*)(r,\varphi) = (D_1 f(\Phi(r,\varphi)) \mid D_2 f(\Phi(r,\varphi)))\, D\Phi(r,\varphi)$$

hence

$$(D_1 f(\Phi(r,\varphi)) \mid D_2 f(\Phi(r,\varphi))) = \left(\frac{\partial}{\partial r}(f^*)(r,\varphi) \mid \frac{\partial}{\partial \varphi}(f^*)(r,\varphi)\right) D\Phi(r,\varphi)^{-1}.$$

Now

$$D\Phi(r,\varphi)^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

This implies

$$\begin{aligned} D_1 f(\Phi(r,\varphi)) &= \left[\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\right](f^*)(r,\varphi), \\ D_2 f(\Phi(r,\varphi)) &= \left[\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial\varphi}\right](f^*)(r,\varphi), \end{aligned}$$

which may be rewritten as the required equalities.

(c) Noting that $D_j f : V \to \mathbb{R}$ is C^1 and applying (b) to $D_j f$, we obtain

$$\begin{aligned} ([D_1^2 f] \circ \Phi)(r,\varphi)) &= [D_1(D_1 f) \circ \Phi](r,\varphi) \\ &= \left[\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\right][D_1 f \circ \Phi](r,\varphi) \\ &= \left[\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\right]^2 f^*(r,\varphi) \\ &= \left[\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\right]\left[\cos\varphi\frac{\partial}{\partial r}\right](f^*)(r,\varphi) \\ &= \left[\cos^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r}\sin^2\varphi\frac{\partial}{\partial r}\right](f^*)(r,\varphi) \end{aligned}$$

Likewise,

$$([D_2^2 f] \circ \Phi)(r,\varphi) = \left[\sin^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cos^2\varphi\frac{\partial}{\partial r}\right](f^*)(r,\varphi)$$

Adding up these identities, we find the desired identity.

Solution 2

(a) By the characterization of submanifolds in the book, there exists an open neighborhood $U_f \ni x^0$ in $\mathbb{R}^n$ and a submersion $\tilde{f} : U_f \to \mathbb{R}^{n-p} = \mathbb{R}^q$ such that $U_f \cap M = \tilde{f}^{-1}(0)$. Since $x^0 \in U_f \cap M \cap U_g \cap N$, we have $\tilde{f}(x^0) = 0$ and $\tilde{g}(x^0) = 0$.

Likewise, we find a submersion $\tilde{g} : U_g \to \mathbb{R}^p$ such that $U_g \cap N = \tilde{g}^{-1}(0)$. Put $U = U_f \cap U_g$, $f = \tilde{f}|_U$ and $g = \tilde{g}|_U$. Then f and g are submersions on U. Furthermore,

$$f^{-1}(0) = U \cap \tilde{f}^{-1}(0) = U \cap U_f \cap M = U \cap M.$$

Likewise $g^{-1}(0) = U \cap N$.

(b) For $x \in U$ we have

$$DF(x) = \begin{pmatrix} Dg(x) \\ Df(x) \end{pmatrix}.$$

From this it is clear that $D(f,g)(x) : \mathbb{R}^n \to \mathbb{R}^{p \times q}$ is a surjective linear map. By the rank theorem from linear algebra it follows that $D(f,g)(x) \in \mathrm{Aut}(\mathbb{R}^n)$.

(c) By the inverse function theorem there exists an open neighborhood $U_0 \ni x^0$ in $\mathbb{R}^n$ such that F maps U_0 diffeomorphically onto an open subset V containing $F(x^0) = 0$. The inverse Φ is a diffeomorphism from V onto U_0. Furthermore, let $x \in U_0$. Then $x \in \Phi(V \cap (\mathbb{R}^p \times \{0\}))$ if and only if $F(x) \in V \cap (\mathbb{R}^p \times \{0\})$, which in turn is equivalent to $F(x) \in V$ and $f(x) = 0$ hence to $x \in \Phi(V)$ and $x \in U \cap M$, which is equivalent to $x \in \Phi(V) \cap M$. The second assertion follows in a similar fashion.

(d) Let $\mathscr{O} = F(V)$, then for $x \in \mathscr{O}$ we have $x \in M \cap N \iff (f(x) = 0$ and $g(x) = 0) \iff F(x) \in (\mathbb{R}^p \times \{0\}) \cap (\{0\} \times \mathbb{R}^q) = \{0\} \iff x \in \Phi(\{0\}) = \{x^0\}$. This establishes the assertion.

(e) The intersection $M \cap N$ is compact. For every $a \in M \cap N$ there exists an open neighborhood $U_a \ni a$ in $\mathbb{R}^n$ such that $U_a \cap M \cap N = \{a\}$. By compactness, there exist finitely many $a_1, \ldots, a_N \in M \cap N$ such that the sets U_{a_j} cover $M \cap N$. Since $U_{a_j} \cap M \cap N = \{a_j\}$, it follows that $M \cap N = \{a_1, \ldots, a_N\}$.

Solution 3

(a) From the assumption it follows that $0 \leq f \leq 1_{\partial B}$. Now B is Jordan measurable, hence ∂B is negligable, and we find

$$0 \leq \underline{\int}_B f(x)\, dx \leq \overline{\int}_B f(x)\, dx \leq \overline{\int}_B 1_{\partial B}(x)\, dx = 0.$$

This implies the assertion.

(b) We observe that $f = 1_B - 1_{B \setminus S}$. The first term is integrable with integral equal to $\mathrm{vol}(B)$; the second is also integrable with zero integral in view of (a). This implies the result.

(c) Fix such $a_1 < u < b_1$ and put $g(v) = f(u,v)$. Then the function $g : I_2 \to \mathbb{R}$ is bounded. Furthermore, if $a_2 < v < b_2$, then $(u,v) \in \mathrm{inw}(B)$ hence $g(v) = f(u,v) = 1$. We see that g equals 1 on (a_2,b_2). This implies that g is Riemann integrable over I_2, with integral equal to $b_2 - a_2$. By definition of Riemann integrability, it follows that lower and upper integral of g over I_2 are equal to each other.

(d) There exists a set $T \subset [a_2,b_2]$ which is not Jordan-measurable. E.g., the set $T := [a_2,b_2] \cap \mathbb{Q}$ has this property. We now take $S = \mathrm{inw}(B) \cup \{b_1\} \times T$. Then $f(b_2,\cdot)$ equals 1_T and is therefore not Riemann integrable. Hence, (c) does not hold for $u = b_1$.

(e) We put $\overline{F}(u)$ for the inner upper integral, and $\underline{F}(u)$. As we argued in (c) we have $\overline{F}(u) = \underline{F}(u) = b_2 - a_2$ for $a_1 < u < b_1$. This means that the functions $\overline{F}$ and $\underline{F}$ are both integrable over $[a_1,b_1]$, with integral $(b_1 - a_1)(b_2 - a_2) = \mathrm{vol}(B)$. This implies the two equalities.

Solution 4

(a) The sets $K_n^\pm$ are closed and bounded in $\mathbb{R}^2$ hence compact. The boundary of $K_n^\pm$ is a finite union of compact subsets of C^1-submanifolds of dimension 1, hence Jordan negligable. It follows that $K_n^\pm$ are Jordan measurable.

(b) We will show this for K_n^+. The other case is treated in a similar fashion. Since K_n^+ is compact Jordan measurable, and f continuous on K_n^+, it follows that f is Riemann-integrable over K_n^+. Hence $1_{K_n^+} f$ is a Riemann integrable function with compact support.

We note that K_n^+ is a compact subset of the open set $\mathbb{R}^2 \setminus L$, where $L = (-\infty, 0] \times \{0\}$. Let $U = (0,\infty) \times (-\pi,\pi)$ and define $\Phi : U \to \mathbb{R}^2$ by $\Phi(r,\varphi) = r(\cos\varphi, \sin\varphi)$. Then Φ is bijective from U onto $\mathbb{R}^2 \setminus L$, and

$$D\Phi(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}.$$

Now $\det D\Phi(r,\varphi) = r > 0$ for $(r,\varphi) \in U$ and we see that Φ is a C^1-diffeomorphism from U onto $\mathbb{R}^2 \setminus L$. We note that $\Phi^{-1}(K_n^+) = [\frac{1}{n}, 1] \times [-\pi/2, \pi/2]$. By application of the substitution of variables theorem, we have

$$\begin{aligned}
\int_{K_n^+} f(x)\,dx &= \int_{\mathbb{R}^2 \setminus L} 1_{K_n^+}(x) f(x)\,dx \\
&= \int_U (1_{K_n^+} f)(\Phi(y)) |\det D\Phi(y)|\,dy \\
&= \int_{-\pi}^{\pi} \int_0^\infty 1_{K_n^+}(\Phi(r,\varphi)) f(\Phi(r,\varphi))\, r\,dr d\varphi \\
&= \int_{-\pi/2}^{\pi/2} \int_{1/n}^1 1 \cdot \frac{1}{r} \cdot r\,dr d\varphi = \pi(1 - \frac{1}{n}).
\end{aligned}$$

(c) We define $K_n = K_n^- \cup K_n^+$. Then K_n is a compact Jordan measurable set. Since K_n^- and K_n^+ overlap on part of their boundaries, hence a negligable set, it follows that $1_{K_n} - 1_{K_n^+} - 1_{K_n^-}$ has Riemann integral zero, so that

$$\int_{\mathbb{R}^2\setminus\{0\}} f(x)\,dx = \int_{K_n^+} f + \int_{K_n^-} f = 2\pi(1-\frac{1}{n}).$$

Taking the limit for $n\to\infty$, we see that $1_{\overline{D}}f$ is absolutely Riemann integrable over $\mathbb{R}^2\setminus\{0\}$ with integral 2π. As ∂D is Jordan negligable and compact, the same holds for $1_D f$. This easily implies that f is absolutely Riemann integrable over $D\setminus\{0\}$ with integral

$$\int_{D\setminus\{0\}} \|x\|^{-1}\,dx = 2\pi.$$