Retake Analyse in Meer Variabelen, WISB212 **2018-07-17, 9:00–12:00**

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Justify your anwers with complete arguments, unless specified otherwise. If you use results from the books or lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 40. The exam grade will be obtained from your total score through division by 4.
- You are free to write the solutions either in English, or in Dutch.

Good Luck !

10 pt total **Exercise 1.** Put $U = \mathbb{R} \times (0, 2\pi)$ and define $\Phi : U \to \mathbb{R}^2$ by $\Phi(t, \varphi) = (e^t \cos\varphi, e^t \sin\varphi)$.

3 pt (a) Calculate $D\Phi(t, \varphi)$ and show that Φ is a C^∞ diffeomorphism onto an open subset V of $\mathbb{R}^2$.

4 pt (b) Let $f : V \to \mathbb{R}$ be a C^1-function. Show that for all $(t, \varphi) \in U$ we have

$$\begin{aligned}
([D_1 f] \circ \Phi)(t, \varphi) &= \left[e^{-t} \cos\varphi \frac{\partial}{\partial t} - e^{-t} \sin\varphi \frac{\partial}{\partial \varphi} \right] (f \circ \Phi)(t, \varphi) \\
([D_2 f] \circ \Phi)(t, \varphi) &= \left[e^{-t} \sin\varphi \frac{\partial}{\partial t} + e^{-t} \cos\varphi \frac{\partial}{\partial \varphi} \right] (f \circ \Phi)(t, \varphi)
\end{aligned}$$

Hint: first calculate $D(f \circ \Phi)(t, \varphi)$.

3 pt (c) If f is C^2 and $t \mapsto f(\Phi(t, \varphi))$ is constant for every $\varphi \in \mathbb{R}$, show that

$$(\Delta f) \circ \Phi = e^{-2t} \frac{\partial^2}{\partial \varphi^2} (f \circ \Phi) \qquad \text{on } U.$$

10 pt total **Exercise 2.** We assume that M is a C^1 submanifold of $\mathbb{R}^n$ of dimension $n-1$ and that $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function. Furthermore, we assume that $Df(x) = 0$ on $T_x M$ for all $x \in M$.

3 pt (a) Show that for every differentiable curve $c : (-1, 1) \to \mathbb{R}^n$ with image contained in M we have $f(c(t)) = f(c(0))$ for all $-1 < t < 1$.

4 pt (b) Show that for every $x^0 \in M$ there exists an open neighborhood W of x^0 in $\mathbb{R}^n$ such that f is constant on $M \cap W$. Hint: use (a) and the definition of submanifold.

3 pt (c) If M is compact show that $f(M)$ is a finite subset of $\mathbb{R}$.

10 pt total **Exercise 3.** We consider the map $\tau : \mathbb{R} \to \mathbb{R}^2$, given by $\tau(\varphi) = 2(\cos\varphi, \sin\varphi)$ and the map $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\Psi(\varphi, \alpha) = ((1 + \tfrac{1}{2}\cos\alpha)\tau(\varphi), \sin\alpha).$$

3 pt (a) Prove that Ψ is an immersion $\mathbb{R}^2 \to \mathbb{R}^3$ with compact image.

1 pt (b) Use a picture to make plausible that the image T of Ψ is a two dimensional torus in $\mathbb{R}^3$. We do not ask for a proof.

2 pt (c) Compute the two dimensional Euclidean area $\mathrm{Area}_2(T)$ of T.

We now consider the subset $M = \Psi([0,\pi] \times [0, 2\pi])$ of T.

1 pt (d) Use a picture to make plausible that M is a two dimensional submanifold with boundary in $\mathbb{R}^3$ whose boundary ∂M consists of the two circles in the plane $x_2 = 0$ with centers $(2,0,0)$ and $(-2,0,0)$ and of radius 1.

3 pt (e) Use Stokes' theorem to calculate the flux through M of the constant vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$ given by $v(x) = e_2 = (0,1,0)^{\mathrm{T}}$.
Hint: first relate v to the vector field $\xi : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto (x_3, 0, -x_1)^{\mathrm{T}}$.

10 pt total **Exercise 4.** We assume that B is a rectangle in $\mathbb{R}^n$, that $U \subset \mathbb{R}^n$ is an open set containing B and that $\varphi : U \to \mathbb{R}$ is a C^1-function with $\varphi(x) > 0$ for all $x \in U$. Finally, we put

$$G = \{(x,t) \in B \times \mathbb{R} \mid x \in B, 0 \le t \le \varphi(x)\}.$$

Let $f : G \to \mathbb{R}$ be a continuous function.

3 pt (a) Show that f is Riemann-integrable over G and that

$$\int_G f(z)\, dz = \int_B \int_0^1 f(x, \varphi(x)t)\varphi(x)\, dt dx.$$

3 pt (b) Show that the map $\Phi : U \times \mathbb{R} \to \mathbb{R}^{n+1}$ given by

$$\Phi(x,t) = (x, \varphi(x)t)$$

is a C^1-diffeomorphism from $U \times \mathbb{R}$ onto an open subset of $\mathbb{R}^{n+1}$.

4 pt (c) By using the substitution of variables theorem in $n+1$ dimensions, show again that the formula of (a) is valid.