Retake Analyse in Meer Variabelen **2018-07-17, 9:00–12:00**

Solution 1

(a) Clearly, Φ is C^∞ and injective. By a simple calculation we have

$$D\Phi(t,\varphi) = \begin{pmatrix} e^t\cos\varphi & -e^t\sin\varphi \\ e^t\sin\varphi & e^t\cos\varphi \end{pmatrix}$$

It now follows that $\det(D\Phi(t,\varphi)) = e^{2t} \neq 0$ for all $(t,\varphi) \in U$. By injectivity of Φ it now follows from the inverse function theorem that Φ is a diffeomorphism from U onto an open subset V of $\mathbb{R}^2$.

(b) We put $f^* = f \circ \Phi$. By the chain rule it follows that for $(t,\varphi) \in U$ we have

$$D(f^*)(t,\varphi) = (D_1 f(\Phi(t,\varphi)) \mid D_2 f(\Phi(t,\varphi)))\, D\Phi(t,\varphi)$$

hence

$$(D_1 f(\Phi(t,\varphi)) \mid D_2 f(\Phi(t,\varphi))) = \left(\frac{\partial}{\partial t}(f^*)(t,\varphi) \mid \frac{\partial}{\partial\varphi}(f^*)(t,\varphi)\right) D\Phi(t,\varphi)^{-1}.$$

Now

$$D\Phi(t,\varphi)^{-1} = e^{-t}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

This implies

$$\begin{aligned} D_1 f(\Phi(t,\varphi)) &= \left[e^{-t}\cos\varphi\frac{\partial}{\partial t} - e^{-t}\sin\varphi\frac{\partial}{\partial\varphi}\right](f^*)(t,\varphi), \\ D_2 f(\Phi(t,\varphi)) &= \left[e^{-t}\sin\varphi\frac{\partial}{\partial t} + e^{-t}\cos\varphi\frac{\partial}{\partial\varphi}\right](f^*)(t,\varphi), \end{aligned}$$

which may be rewritten as the required equalities.

(c) Noting that $D_1 f : V \to \mathbb{R}$ is C^1 and applying (b) to $D_1 f$, we obtain

$$\begin{aligned} ([D_1^2 f]\circ\Phi)(t,\varphi)) &= [D_1(D_1 f)\circ\Phi](t,\varphi) \\ &= \left[e^{-t}\cos\varphi\frac{\partial}{\partial t} - e^{-t}\sin\varphi\frac{\partial}{\partial\varphi}\right][D_1 f\circ\Phi](t,\varphi) \\ &= \left[e^{-t}\cos\varphi\frac{\partial}{\partial t} - e^{-t}\sin\varphi\frac{\partial}{\partial\varphi}\right]^2 f^*(t,\varphi) \\ &= \left[e^{-t}\cos\varphi\frac{\partial}{\partial t} - e^{-t}\sin\varphi\frac{\partial}{\partial\varphi}\right]\left[-e^{-t}\sin\varphi\frac{\partial}{\partial\varphi}\right](f^*)(t,\varphi) \\ &= \left[2e^{-2t}\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi} + e^{-2t}\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}\right](f^*)(t,\varphi) \end{aligned}$$

Likewise,

$$([D_2^2 f]\circ\Phi)(r,\varphi) = \left[-2e^{-2t}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + e^{-2t}\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\right](f^*)(t,\varphi)$$

Adding up these identities, we find the desired identity.

Solution 2

(a) Let c be such a differentiable curve then $c'(t) \in T_{c(t)}M$ by definition of the tangent space. By the chain rule we now have

$$\frac{d}{dt} f(c(t)) = Df(c(t))c'(t) = 0,$$

for all $t \in (-1,1)$. Since $f \circ c : (-1,1) \to \mathbb{R}$ is differentiable, it follows that $f(c(t)) = f(c(0))$ for all $-1 < t < 1$.

(b) Since M is a submanifold, there exists an open neighborhood W^0 of x^0 in $\mathbb{R}^n$ and a diffeomorphism Φ from W^0 onto an open neighborhood V^0 of 0 in $\mathbb{R}^n$ with $\Phi(x^0) = 0$ and $\Phi(W^0 \cap M) = V^0 \cap (\mathbb{R}^{n-1} \times \{0\})$. We may fix $\delta > 0$ such that $V := (-\delta, \delta)^n$ is contained in V^0. We note that $V \cap \Phi(W^0 \cap M) = (-\delta, \delta)^{n-1} \times \{0\}$. Let $y \in V \cap \Phi(W^0 \cap M)$. Then $y_n = 0$ and $d : t \mapsto ty$, $(-1,1) \to V$ is a differentiable curve in V which is contained in $\Phi(W^0 \cap M)$.

By (a) it follows that f is constant along $\Phi^{-1} \circ d$. Hence $f \circ \Phi^{-1}$ is constant on $(-1,1)y$. Since this is true for all y, the function $f \circ \Phi^{-1}$ is constant on $V \cap \Phi(W^0 \cap M)$. This implies that f is constant on $\Phi^{-1}(V) \cap W^0 \cap M = \Phi^{-1}(V) \cap M$.

Put $W = \Phi^{-1}(V)$, then we see that W is an open neighborhood of x^0 in $\mathbb{R}^n$ and f is constant on $W \cap M$.

(c) For every $x \in M$ there exists an open neighborhood W_x of x in $\mathbb{R}^n$ such that f is constant on $W_x \cap M$. By compactness there exists a finite collection of points $x_1, \ldots, x_N \in M$ such that $M \subset \cup_j (W_{x_j} \cap M)$. It follows that $f(M) \subset \cup_j f(W_{x_j} \cap M) \subset \{f(x_1), \ldots, f(x_N)\}$, which is finite.

Solution 3

(a) We calculate

$$D_1\Psi(\varphi, \alpha) = ((1 + \tfrac{1}{2}\cos\alpha)\tau'(\varphi), 0)^{\mathrm{T}}$$

and

$$D_2\Psi(\varphi, \alpha)(-\sin\alpha\ \tau(\varphi), \cos\alpha)^{\mathrm{T}}$$

Accordingly,

$$D_1\Psi(\varphi, \alpha) \times D_2\Psi(\varphi, \alpha) = \begin{pmatrix} \cos\alpha(1 + \frac{1}{2}\cos\alpha)\tau_2'(\varphi) \\ -\cos\alpha(1 + \frac{1}{2}\cos\alpha)\tau_1'(\varphi) \\ (1 + \frac{1}{2}\cos\alpha)\sin\alpha \end{pmatrix} \qquad (*)$$

The length of this vector is given by

$$\|D_1\Psi(\varphi, \alpha) \times D_2\Psi(\varphi, \alpha)\| = 2(1 + \frac{1}{2}\cos\alpha).$$

If follows that this length is nowhere zero. Hence $D\Psi(\varphi, \alpha)$ is injective for all φ, α and we conclude that $D\Psi(\varphi, \alpha)$ is an immersion. The image of Ψ is the image of $\Psi([0, 2\pi] \times [0, 2\pi])$ which is compact, since $[0, 2\pi] \times [0, 2\pi]$ is compact and Ψ is continuous.

(b) Observe that $\varphi \mapsto (\tau(\varphi), 0)$ parametrizes a circle C in $x_3 = 0$ of center 0 and radius 2. Next write

$$\Psi(\varphi, \alpha) = (\tau(\varphi), 0) + (\cos\alpha \frac{\tau(\varphi)}{2}, \sin\alpha)$$

to see that T consists of the points in $\mathbb{R}^3$ of distance 1 to the circle C.

(c) Since Ψ is injective on $[0,2\pi)\times[0,2\pi)$ with image T, the area is calculated by

$$\begin{aligned}\text{Area}_2(T) &= \int_0^{2\pi}\int_0^{2\pi}\|D_1\Psi(\varphi,\alpha)\times D_2\Psi(\varphi,\alpha)\|d\varphi d\alpha \\ &= 2\pi\int_0^{2\pi}2(1+\frac{1}{2}\cos\alpha)\,d\alpha = 8\pi^2.\end{aligned}$$

(d) The boundary is given by $\partial M = C_1\cup C_2$, where $C_1 = \Psi(\{0\}\times[0,2\pi])$ and $C_2 = \Psi(\{\pi\}\times[0,2\pi])$ are two circles of radius 1 in the plane $x_2 = 0$. The centers of these circles are $(2,0,0)$ and $(-2,0,0)$, respectively.

(e) By an easy calculation it follows that $\text{curl}(\xi) = 2v$. We equip M with the orientation determined by the normal (*). The associated unit normal is denoted by $\mathbf{n}$. It follows from Stokes' theorem that the flux of v through M relative to this choice of normal is given by

$$\int_M v(x)\cdot\mathbf{n}(x)\,d_2x = \frac{1}{2}\int_{\partial M}\xi(x)\cdot\mathbf{e}(x)\,d_1x.$$

Here $\mathbf{e}(x)$ denotes the positively oriented unit tangent vector to ∂M at the point $x\in\partial M$. We will proceed by computing the right-hand side, using that ∂M is the disjoint union the two circles C_1, C_2.

The circle C_1 is parametrized by $\gamma_1:\alpha\mapsto\Psi(0,\alpha) = (2+\cos\alpha, 0, \sin\alpha)$ with $0\le\alpha\le 2\pi$. Thus, $\gamma_1'(\alpha) = D_2\Phi(0,\alpha) = (-\sin\alpha, 0, \cos\alpha)$, which has unit length, so that $\mathbf{e}(\gamma_1(\alpha)) = \pm\gamma_1'(\alpha)$. Now $-D_1\Phi(0,\alpha)$ is tangent to M and normal to C_1 in the outward direction. Since the basis $-D_1\Phi(0,\alpha)$, $D_2\Phi(0,\alpha)$, $\mathbf{n}(\Phi(0,\alpha))$ is negatively oriented, it follows that the minus sign should be taken in $\pm$. Hence,

$$\frac{1}{2}\int_{C_1}\xi(x)\cdot\mathbf{e}(x)\,d_1x = -\frac{1}{2}\int_0^{2\pi}\xi(\gamma_1(\alpha))\cdot\gamma_1'(\alpha)\,d\alpha = \frac{1}{2}\int_0^{2\pi}(1+2\cos\alpha)\,d\alpha = \pi.$$

On the other hand, C_2 is parametrized by $\gamma_2:\alpha\mapsto\Psi(\pi,\alpha) = (-2-\cos\alpha, 0, \sin\alpha)$ with $0\le\alpha\le 2\pi$. Again $\mathbf{e}(\gamma_2(\alpha)) = \pm\gamma_2'(\alpha)$. This time, $D_1\Psi(0,\alpha)$ is tangent to M and outward normal to C_2 so that the plus sign should be taken. Hence,

$$\frac{1}{2}\int_{C_2}\xi(x)\cdot\mathbf{e}(x)\,d_1x = \frac{1}{2}\int_0^{2\pi}\xi(\gamma_2(\alpha))\cdot\gamma_2'(\alpha)\,d\alpha = \frac{1}{2}\int_0^{2\pi}(1+2\cos\alpha)\,d\alpha = \pi.$$

We conclude that the flux of v through M relative to $\mathbf{n}$ equals $\pi+\pi = 2\pi$.

Remark. The following solution is also allowed. By a second application of Stokes theorem, this time to the disks D_1 and D_2 with boundaries C_1 and C_2, it follows that

$$\frac{1}{2}\int_{C_j}\xi(x)\cdot\mathbf{e}(x)\,d_1x = \int_{D_j}v(x)\cdot\mathbf{n}_j(x)\,d_2x.$$

Here the normal should be taken in accordance with the orientation of C_j. In both cases, $\mathbf{n}_j = (0,1,0) = v$ and we see that

$$\int_{D_j}v(x)\cdot\mathbf{n}_j(x)\,d_2x = \int_{D_j}d_2x = \text{Area}_2(D_j) = \pi.$$

Thus, again, the flux is seen to be equal to $\pi+\pi = 2\pi$.

Solution 4

(a) Write $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Since the function φ is continuous and non-negative it follows by application of Thm 6.4.5 in the book that f is integrable over G and that

$$\begin{aligned}\int_G f(z)dz &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_0^{\varphi(x)} f(x,s)\, ds\, dx_n \cdots dx_1 \\ &= \int_B \int_0^{\varphi(x)} f(x,s)\, ds\, dx \\ &= \int_B \int_0^1 f(x, \varphi(x)t)\varphi(x)\, dt\, dx;\end{aligned}$$

in the last step we applied the substitution $s = \varphi(x)t$ to the inner integral. In the above, it is OK if the first step of the derivation is taken for granted.

(b) Since φ is everywhere on U strictly positive, it readily follows that the defined map Φ is injective. By substitution of variables, it is C^1. Furthermore, for $1 \leq j \leq n$ we have

$$D_j\Phi(x,t) = (e_j, D_j\varphi(x)t)^{\mathrm{T}}$$

and

$$D_{n+1}\Phi(x,t) = (0, \varphi(x))^{\mathrm{T}}.$$

It follows that

$$D\Phi(x,t) = \begin{pmatrix} I & 0 \\ D\varphi(x)t & \varphi(x) \end{pmatrix}.$$

Hence, $\det D\Phi(x,t) = \varphi(x) > 0$ for $(x,t) \in U$ and by the inverse function theorem it follows that Φ is a diffeomorphism as stated.

(c) It is readily verified that $G = \Phi(B \times I)$. Outside G we may extend f by zero. Then f is a compactly supported Riemann integrable function on $\Phi(U \times \mathbb{R})$.

By substitution of variables we have that

$$\begin{aligned}\int_G f(z)\, dz &= \int_{\Phi(U\times\mathbb{R})} f(z)\, dz \\ &= \int_{U\times\mathbb{R}} f(\Phi(y))\, |\det D\Phi(y)|\, dy \\ &= \int_{B\times[0,1]} f(\Phi(y))\, |\det D\Phi(y)| dy \\ &= \int_B \int_0^1 f(\Phi(x,t))\, |\det D\Phi(x,t)| dt dx \\ &= \int_B \int_0^1 f(x, \varphi(x)t)\, \varphi(x)\, dx.\end{aligned}$$