Solution of Exercise 0.1

(i) We have

$$p(x,y) = x^2 + 2y_1x + y_1^2 - (y_1^2 - y_2) = (x + y_1)^2 - \Delta(y).$$

If $p(x,y) = 0$ then the assertion of $(\star)$ is obvious as squares are nonnegative. It follows that every solution $x \in \mathbf{R}$ of $p(x,y) = 0$ is given by $x_\pm = -y_1 \pm \sqrt{\Delta(y)}$; accordingly, maximally two do exist. Obviously $x_+ = x_-$ if and only if $\Delta(y) = 0$; hence, the final assertion is a direct consequence of $(\star)$.

(ii) The equality $p(x,y) = 0$ is equivalent with $y_2 = -x^2 - 2y_1x$, which shows that $N = \mathrm{im}(\phi)$. Furthermore, $N = \mathrm{graph}(f)$ where $f : \mathbf{R}^2 \to \mathbf{R}$ with $f(x,y_1) = -x^2 - 2y_1x$ is a C^∞ function; therefore N is a C^∞ submanifold of $\mathbf{R}^3$ of dimension 2 on the basis of Definition 4.2.1.

(iii) The identity $Dp(x,y) = (*,*,1)$ shows that the rank of $Dp(x,y)$ equals 1 everywhere; in other words, $Dp(x,y)$ is surjective, for all $(x,y) \in \mathbf{R}^3$. Hence the second assertion is a direct consequence of the Submersion Theorem 4.5.2.

(iv) Differentiation immediately yields the following formulae:

$$D\Phi(x,y_1) = \begin{pmatrix} 0 & 1 \\ -2x - 2y_1 & -2x \end{pmatrix} \qquad \text{and} \qquad \det D\Phi(x,y_1) = 2(x + y_1).$$

By definition, the determinant vanishes at singular points. Hence, the identification of the set of singular points with S follows directly, whereas the equation above obviously is that of a straight line. The assertion on the rank of $D\Phi(x,y_1)$, for $(x,y_1) \in S$, follows from the fact that in this case

$$D\Phi(x,y_1) = \begin{pmatrix} 0 & 1 \\ 0 & * \end{pmatrix}.$$

(v) Suppose $(x,y) \in \mathbf{R}^3$ satisfies $\Phi(x,y_1) = y$. Then, in particular, we have $p(x,y) = 0$ and so we obtain from $(\star)$ in part (i) that $\Delta(y) \geq 0$. Hence the inclusions $\Phi(S) \subset P$ and $\Phi(\mathbf{R}^2 \setminus S) \subset \{\, y \in \mathbf{R}^2 \mid \Delta(y) > 0 \,\}$ are obvious on the basis of $(\star)$ again. Now we prove the reverse inclusions. According to part (i) the condition $\Delta(y) = 0$ on $y \in \mathbf{R}^2$ ensures that there is a unique solution $x \in \mathbf{R}$ for $p(x,y) = 0$, i.e., $y = \Phi(x,y_1)$; furthermore, $(\star)$ then implies that $(x,y_1) \in S$. Next, suppose $y \in \mathbf{R}^2$ satisfies $\Delta(y) > 0$. From part (i) we then obtain the existence of two different solutions $x_\pm$ of the equation $p(x,y) = 0$, and this gives two distinct elements $(x_\pm, y_1) \in \mathbf{R}^2$ both belonging to $\Phi^{-1}(\{y\})$. Using $(\star)$ once more, we actually get $(x_\pm, y_1) \in \mathbf{R}^2 \setminus S$. In geometric terms, lines in $\mathbf{R}^3$ parallel to the x-axis, which means being of the form $\{\, (x,y) \in \mathbf{R}^3 \mid x \in \mathbf{R} \,\}$, intersect the surface N once, and twice, if $\Delta(y)$ is 0, and positive, respectively, and in no other case.

(vi) By definition $\Phi = \pi \circ \phi$; hence, we obtain $\pi^{-1} \circ \Phi = \phi$ (abusing the notation for the inverse image). Application of this identity to the set S gives the equality $\phi(S) = \pi^{-1}(P)$. Next, suppose $(x,y_1) \in S$, in other words, $y_1 = -x$. Then $\phi(x,y_1) = (x, -x, y_2) \in \phi(S) = \pi^{-1}(P)$ implies $y_2 = x^2$. Accordingly

$$\phi(x,y_1) = (x, -x, x^2) = \sigma(x), \qquad \text{that is,} \qquad \phi(S) \subset \Sigma.$$

Conversely, $(x,y) \in \Sigma$ implies

$$(x,y) = \sigma(x) = (x, -x, x^2) = \phi(x, -x), \qquad \text{i.e.,} \qquad \Sigma \subset \phi(S).$$

Now the last assertion. $(x,y) \in \Sigma$ means that x is a solution of $p(X,y) = (X - x)^2 = X^2 - 2xX + x^2 = 0$, and as a consequence x is a solution of $D_1p(X,y) = 2(X - x)$ too. Accordingly, $p(x,y) = D_1p(x,y) = 0$. Conversely, suppose $(x,y) \in \mathbf{R}^3$ satisfies $p(x,y) = 0$ and $D_1p(x,y) = 2(x + y_1) = 0$; hence, in particular, $y_1 = -x$. Hence $(x,y) \in \phi(S) = \Sigma$.

(vii) If y_1 is fixed and $p(x, y) = 0$, we get from $(\star)$ in part (i)

$$y_2 = y_1^2 - \Delta(y) = y_1^2 - (x + y_1)^2.$$

The right-hand side is maximal if $x + y_1 = 0$ and if this is the case it assumes the value y_1^2. Hence the vertex of the parabola has coordinates $(-y_1, y_1, y_1^2) = \sigma(-y_1)$ and it also opens downward.

(viii) In view of $D\sigma(x) = (1, -1, 2x)$, a parametric representation for $\Lambda(x)$ is given by $\sigma(x) + \mathbf{R}(1, -1, 2x)$.

(ix) $(0, -1, 2x)$ is the orthogonal projection of $D\sigma(x)$ onto the (y_1, y_2)-plane along the x-axis; hence, $N(x)$ may be described as given. By definition, the lines $N(x)$ are disjoint, for distinct $x \in \mathbf{R}$. Furthermore, consider $(x, y) \in N(x)$, that is, satisfying $y_1 = -x - \lambda$ and $y_2 = x^2 + 2\lambda x$, for some $\lambda \in \mathbf{R}$. Then $(x, y) \in N$ as follows from

$$p(x, y) = x^2 + 2y_1 x + y_2 = x^2 - 2(x + \lambda)x + x^2 + 2\lambda x = 0.$$

Accordingly, every $N(x)$ is contained in N. Conversely, suppose $x \in \mathbf{R}$ is fixed and $(x, y) \in \mathbf{R}^3$ belongs to N. Then there exists $\lambda \in \mathbf{R}$ such that $y_1 = -x - \lambda$, while $p(x, y) = 0$ now implies

$$y_2 = -x^2 - 2y_1 x = x^2 + 2\lambda x; \qquad \text{i.e.,} \qquad (x, y) \in N(x).$$

The equality $N(x) = \sigma(x) + \mathbf{R}(0, -1, 2x)$ implies that $N(x)$ intersects Σ in $\sigma(x)$, and this is the only point of intersection as the elements of Σ are uniquely determined by their first component.