Solution of Exercise 0.1

(i) $x \in V$ implies $0 \leq x_1^2 + x_2^2 = (1 - x_3)x_3^2$, therefore $0 \leq 1 - x_3$, that is, $x_3 \leq 1$.

(ii) If $x_3 = 1 - s^2$, then $1 - x_3 = s^2$. Accordingly, for $x \in V$,

$$x_1^2 + x_2^2 = (1 - x_3)x_3^2 = \big(s(1 - s^2)\big)^2, \qquad \text{so} \qquad (x_1, x_2) = s(1 - s^2)(\cos t, \sin t),$$

for suitable $t \in \mathbf{R}$, on account of the parametrization of a circle by trigonometric functions. Thus we obtain $V \subset \mathrm{im}(\phi)$. Conversely, for every $x \in \mathrm{im}(\phi)$,

$$x_1^2 + x_2^2 = \big(s(1 - s^2)\big)^2 \qquad \text{and} \qquad (1 - x_3)x_3^2 = s^2(1 - s^2)^2, \qquad \text{that is} \qquad g(x) = 0.$$

(iii) Suppose, for $h \in \mathbf{R}^2$,

$$D\phi(s, t)h = \begin{pmatrix} (1 - 3s^2)\cos t & -s(1 - s^2)\sin t \\ (1 - 3s^2)\sin t & s(1 - s^2)\cos t \\ -2s & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ -2sh_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $s \neq 0$, it follows that $h_1 = 0$. The two top equations above then give

$$h_2 s(1 - s^2)\sin t = h_2 s(1 - s^2)\cos t = 0, \qquad \text{so} \qquad s(1 - s^2)h_2 = 0.$$

Accordingly, if $s \notin \{-1, 0, 1\}$, then $h_2 = 0$ too; and therefore ϕ is immersive in this case. On the other hand,

$$D\phi(\pm 1, t) = -2 \begin{pmatrix} \cos t & 0 \\ \sin t & 0 \\ \pm 1 & 0 \end{pmatrix}, \qquad D\phi(0, t) = \begin{pmatrix} \cos t & 0 \\ \sin t & 0 \\ 0 & 0 \end{pmatrix},$$

which shows that all three of these mappings in $\mathrm{Lin}(\mathbf{R}^2, \mathbf{R}^3)$ have a one-dimensional kernel. It is direct from the definition that $\phi(\pm 1, t) = 0$ and $\phi(0, t) = n$, for all $t \in \mathbf{R}$.

(iv) We have, for $x \in V$,

$$Dg(x) = (2x_1, 2x_2, -2x_3 + 3x_3^2) \in \mathrm{Lin}(\mathbf{R}^3, \mathbf{R}).$$

This mapping fails to be surjective only if all its entries equal 0, which is the case only if $x = 0$ (the solution with $x_3 = \frac{2}{3}$ does not belong to V). Hence, g is submersive at all points of $V \setminus \{0\}$; and on the strength of the Submersion Theorem we now obtain that V is a C^∞ manifold in $\mathbf{R}^3$ of dimension 2 at all of its points, with the possible exception of the point 0.

Solution of Exercise 0.2

(i) $x^2 + 2y_1 x + y_2 = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2$ implies $y_1 = -\frac{1}{2}(x_1 + x_2)$ and $y_2 = x_1 x_2$.

(ii) The coefficients of $\Phi(x)$ are symmetric in x_1 and x_2. Horizontal lines are of the form $\{x \in \mathbf{R}^2 \mid x_2 = \text{constant}\}$.

(iii) Suppose $y = \Phi(x)$, that is, $2y_1 = -x_1 - x_2$ and $y_2 = x_1 x_2$. Then

$$2x_2 y_1 = -x_1 x_2 - x_2^2 = -y_2 - x_2^2, \qquad \text{so} \qquad p(x_2, y) = 0, \qquad \text{that is} \qquad y_2 = -2x_2 y_1 - x_2^2;$$

and this shows that y belongs to the straight line $L(x_2)$ in $\mathbf{R}^2$ of slope $-2x_2$.

(iv) We have

$$D\Phi(x) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ x_2 & x_1 \end{pmatrix}, \qquad \det D\Phi(x) = -\frac{1}{2}(x_1 - x_2) = 0 \qquad \Longrightarrow \qquad x_1 = x_2.$$

Hence $S = \{ (x_2, x_2) \in \mathbf{R}^2 \mid x_2 \in \mathbf{R} \}$, the diagonal in $\mathbf{R}^2$. Now $(y_1, y_2) = \Phi(x_2, x_2) = (-x_2, x_2^2)$ satisfies $y_1^2 = x_2^2 = y_2$, which implies

$$P = \Phi(S) \subset \{ y \in \mathbf{R}^2 \mid y_1^2 - y_2 = 0 \} =: \widetilde{P}.$$

Conversely, if $y_1^2 = y_2$, then we have $y_2 \geq 0$; hence there exists $x_2 \in \mathbf{R}$ satisfying $y_2 = x_2^2$. Then $y_1^2 = x_2^2$, having a solution $y_1 = -x_2$, that is, $y = \Phi(x_2, x_2)$. It follows that $\widetilde{P} \subset P$ and therefore $P = \widetilde{P}$.

(v) Indeed, given $y \in V$, the system of equations $x_1 + x_2 = -2y_1$ and $x_1 x_2 = y_2$ for $x \in \mathbf{R}^2$ is equivalent to the system $x_1^2 + 2y_1 x_1 + y_2 = 0$ and $x_2 = -x_1 - 2y_1$. The latter system has a solution $x \in \mathbf{R}^2 \setminus S$, because $y \in V$ represents the well-known discriminant criterion for $p(X, y)$ having two distinct real roots. Hence $y = \Phi(x)$, and therefore $y \in L(x_1) \cap L(x_2)$ with $x_1 \neq x_2$.

(vi) Consider $y \in L(x_2) \cap P$. According to part (iv) the condition $y \in P$ implies the existence of $\widetilde{x}_2 \in \mathbf{R}$ such that $y = \Phi(\widetilde{x}_2, \widetilde{x}_2)$. Furthermore, the condition $y \in L(x_2)$ now gives

$$0 = x_2^2 - 2x_2\widetilde{x}_2 + \widetilde{x}_2^2 = (x_2 - \widetilde{x}_2)^2, \qquad \text{so} \qquad x_2 = \widetilde{x}_2, \qquad \text{hence} \qquad y = \Phi(x_2, x_2).$$

The tangent line of P at $\Phi(x_2, x_2)$ is the set of $y \in \mathbf{R}^2$ satisfying

$$(2y_1, -1)\Big|_{y = (-x_2, x_2^2)} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -(2x_2 y_1 + y_2) = 0.$$

As a consequence, the geometric tangent line of P at $\Phi(x_2, x_2)$ equals $\{ y \in \mathbf{R}^2 \mid 2x_2 y_1 + y_2 = c \}$ where $c \in \mathbf{R}$ is determined by $c = -2x_2^2 + x_2^2 = -x_2^2$; in other words, the geometric tangent line equals $L(x_2)$.

(vii) Obvious.