Solution of Exercise 0.1

(i) We have $\operatorname{curl} f(x) = 1$, for all $x \in \mathbf{R}^2$, hence Green's Integral Theorem 8.3.5 implies

$$\text{oppervlakte}(\Omega) = \int_\Omega \operatorname{curl} f(x)\,dx = \int_{\partial\Omega} \langle\, f(y), d_1 y\,\rangle = \sum_{1\le k\le n} \int_{\partial\Omega_k} \langle\, f(y), d_1 y\,\rangle,$$

$$\text{where} \qquad \partial\Omega_k = \{\, y^{(k)}(t) := x^{(k)} + t\,(x^{(k+1)} - x^{(k)}) \in \mathbf{R}^2 \mid 0 \le t \le 1\,\}.$$

As a consequence,

$$f \circ y^{(k)}(t) = \big(0, x_1^{(k)} + t\,(x_1^{(k+1)} - x_1^{(k)})\big), \qquad D y^{(k)}(t) = x^{(k+1)} - x^{(k)},$$

$$\langle\, f \circ y^{(k)}(t), D y^{(k)}(t)\,\rangle = (x_2^{(k+1)} - x_2^{(k)})(x_1^{(k)} + t\,(x_1^{(k+1)} - x_1^{(k)})),$$

$$\int_{\partial\Omega_k} \langle\, f(y), d_1 y\,\rangle = (x_2^{(k+1)} - x_2^{(k)}) \int_0^1 (x_1^{(k)} + t\,(x_1^{(k+1)} - x_1^{(k)}))\,dt$$

$$= \big(x_2^{(k+1)} - x_2^{(k)}\big)\big(x_1^{(k)} + \frac{1}{2}(x_1^{(k+1)} - x_1^{(k)})\big) = \frac{1}{2}\big(x_1^{(k+1)} + x_1^{(k)}\big)\big(x_2^{(k+1)} - x_2^{(k)}\big).$$

(ii) Write Ω as a union of n triangles with vertices 0, $x_1^{(k)}$ and $x_1^{(k+1)}$, for $1 \le k \le n$. Next, note that the area of such a triangle equals half the area of the parallelogram spanned by the vectors $x_1^{(k)}$ and $x_1^{(k+1)}$, where the latter area is given by

$$\begin{vmatrix} x_1^{(k)} & x_1^{(k+1)} \\ x_2^{(k)} & x_2^{(k+1)} \end{vmatrix} = x_1^{(k)} x_2^{(k+1)} - x_1^{(k+1)} x_2^{(k)}.$$

For another proof of the identity in part (ii), expand the products at its left–hand side and observe that

$$\sum_{1\le k\le n} x_1^{(k+1)} x_2^{(k+1)} - \sum_{1\le k\le n} x_1^{(k)} x_2^{(k)} = 0.$$

Solution of Exercise 0.2

(i) Suppose $x \in S$, then $x_1^2 + x_2^2 = 1 - x_3^2 \ne 0$. Furthermore, $\ell_x = \{\, (\lambda x_1, \lambda x_2, x_3) \mid \lambda \in \mathbf{R}\,\}$. Now

$$(\lambda x_1, \lambda x_2, x_3) \in C^2 \qquad \Longrightarrow \qquad \lambda^2 (x_1^2 + x_2^2) = 1 \qquad \Longrightarrow \qquad \lambda = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}}.$$

The point of intersection of ℓ_x with C^2 closest to x is obtained by taking the plus sign. This proves the formula for f. Furthermore, given arbitrary $y \in C^2$, an element $x \in S$ such that $f(x) = y$ has to satisfy

$$x_3 = y_3, \qquad \Longrightarrow \qquad \sqrt{x_1^2 + x_2^2} = \sqrt{1 - x_3^2} = \sqrt{1 - y_3^2}, \qquad \Longrightarrow \qquad x_j = y_j \sqrt{1 - y_3^2},$$

for $1 \le j \le 2$. Indeed, such an x belongs to S, in view of

$$\|x\|^2 = (y_1^2 + y_2^2)(1 - y_3^2) + y_3^2 = 1.$$

As a consequence, $x \in S$ exists and is uniquely determined. This establishes the bijectivity of f and also that

$$f^{-1}(y) = (y_1\sqrt{1-y_3^2},\, y_2\sqrt{1-y_3^2},\, y_3).$$

(ii) As is well-known, up to subsets of negligible area, two-dimensional submanifolds V contained in S are of the form $V = \phi(D)$, with $\phi : D \to S^2$ given by

$$D \subset \,]-\pi, \pi\,[\, \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\qquad \text{and} \qquad \phi(\alpha, \theta) = (\cos\alpha\cos\theta, \sin\alpha\cos\theta, \sin\theta).$$

Note that we may take D to be open and that ϕ is an embedding. As in Example 7.4.6 we see

$$\text{oppervlakte}(V) = \int_D \cos\theta \, d\alpha d\theta.$$

On account of f and f^{-1} being a differentiable bijections (on suitable open subsets of $\mathbf{R}^3$) we see that $\widetilde{\phi} = f \circ \phi : D \to C^2$ is an embedding, which is given by

$$\widetilde{\phi}(\alpha, \theta) = (\cos\alpha, \sin\alpha, \sin\theta).$$

$f(V) = \widetilde{\phi}(D)$ is a submanifold in $\mathbf{R}^3$ of dimension 2 that is contained in C^2 because of Corollary 4.3.2. Furthermore,

$$\frac{\partial\widetilde{\phi}}{\partial\alpha}(\alpha, \theta) = (-\sin\alpha, \cos\alpha, 0), \qquad \frac{\partial\widetilde{\phi}}{\partial\theta}(\alpha, \theta) = (0, 0, \cos\theta),$$

$$\frac{\partial\widetilde{\phi}}{\partial\alpha} \times \frac{\partial\widetilde{\phi}}{\partial\theta}(\alpha, \theta) = \cos\theta(\cos\alpha, \sin\alpha, 0), \qquad \left\|\frac{\partial\widetilde{\phi}}{\partial\alpha} \times \frac{\partial\widetilde{\phi}}{\partial\theta}(\alpha, \theta)\right\| = \cos\theta.$$

Therefore $f(V) = \widetilde{\phi}(D)$ implies

$$\text{oppervlakte}\,(f(V)) = \int_D \cos\theta \, d\alpha d\theta.$$

(iii) The assertion is a direct consequence of Exercise 3.6 on cylindrical coordinates.

(iv) If $W \subset C^2$, then $W = \psi(D)$, for some D as in part (ii), while

$$\frac{\partial\psi}{\partial\alpha}(\alpha, x_3) = (-\sin\alpha, \cos\alpha, 0), \qquad \frac{\partial\psi}{\partial x_3}(\alpha, x_3) = (0, 0, 1),$$

$$\frac{\partial\psi}{\partial\alpha} \times \frac{\partial\psi}{\partial x_3}(\alpha, x_3) = (\cos\alpha, \sin\alpha, 0), \qquad \left\|\frac{\partial\psi}{\partial\alpha} \times \frac{\partial\psi}{\partial x_3}(\alpha, x_3)\right\| = 1,$$

This and the fact that $g(W) = D$ now yield

$$\text{oppervlakte}(W) = \int_D d\alpha dx_3 = \text{oppervlakte}\,(g(W)).$$

(v) We may assume that the great circles intersect at the poles of S^2, since this can be achieved by applying a rotation of $\mathbf{R}^3$, which is area-preserving. Now the image $f(V)$ is a curved rectangle on C^2 of width α and height 2. Next unroll S^2 on the plane $\mathbf{R}^2$, in other words, apply g. Then the curved rectangle will be mapped to a genuine rectangle in $\mathbf{R}^2$ of width α and height 2. Application of parts (ii) and (iv) now yields that the area of V equals 2α. In particular, S^2 is the spherical diangle of angle 2π, which implies that its area is 4π.

Solution of Exercise 0.3

(i) The assertion follows from that fact that $f|_V$ is a continuous function on the compact set V. Furthermore, E consists of more points than the origin alone.

(ii) The method of Lagrange multipliers gives that $f|_V$ is extremal at points $x \in V$ for which there exist $\lambda \in \mathbf{R}^2$ satisfying

$$(*) \qquad 2x_j = 2\lambda_1 a_j x_j + \lambda_2 p_j \qquad (1 \le j \le 3).$$

Taking the inner product with x we obtain

$$2\|x\|^2 = 2\lambda_1 \sum_{1\le j\le 3} a_j x_j^2 + \lambda_2 \langle p, x\rangle = 2\lambda_1, \qquad \text{hence} \qquad m = \|x\|^2 = \lambda_1.$$

Consider first the case of $\lambda_2 = 0$. Then $(*)$ implies

$$x_j(1 - ma_j) = 0 \qquad (1 \le j \le 3).$$

If two of the coordinates of x are equal to 0, then the condition $x \in P$ forces the third coordinate to be 0, but then $x \notin E$. Consequently two of these coordinates differ from 0, which implies $0 = 1 - ma_i = 1 - ma_j$, for $i \ne j$; that is, $a_i = a_j$, a contradiction. Hence $\lambda_2 \ne 0$. If $ma_j - 1 = 0$, for some j, then $(*)$ implies $\lambda_2 p_j = 0$; and thus $p_j = 0$, a contradiction. Furthermore, we obtain from $(*)$

$$x_j = \frac{\lambda_2}{2} \frac{p_j}{1 - ma_j} \qquad (1 \le j \le 3).$$

But now $x \in P$ gives

$$\frac{\lambda_2}{2} \sum_{1\le j\le 3} \frac{p_j^2}{1 - ma_j} = 0, \qquad \text{and so} \qquad \sum_{1\le j\le 3} \frac{p_j^2}{ma_j - 1} = 0.$$