Mathematisch Instituut, Faculteit Wiskunde en Informatica, UU.
In elektronische vorm beschikbaar gemaakt door de $\mathcal{T}_{\!\mathcal{B}}\mathcal{C}$ van A–Eskwadraat.
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# Uitwerking[1] Analyse in Meer Variabelen (WISB212) 2007-04-17

**Exercise 0.1**

(i) The function $\sqrt{\cdot}\ : ]0,\infty[ \to \mathbb{R}$ is of class $C^\infty$. Hence, $f$ is the composition of $C^\infty$ functions, therefore the assertion follows from the chain rule.

(ii) $f^2(x) = \langle x, x\rangle$ implies $Df^2(x)h = 2\langle Ax, Ah\rangle$ according ot Corollary 2.4.3.(ii). Hence the desired formula follows from $2f(x)Df(x)h = Df^2(x)h = 2\langle Ax, Ah\rangle$ on account of the chain rule. Furthermore

$$Df(x)h = \frac{\langle Ax, Ah\rangle}{f(x)} = \frac{1}{f(x)}(Ax)^t Ah = \frac{1}{f(x)} x^t A^t Ah.$$

(iii) We have

$$D_j f(x) = Df(x)e_j = \frac{\langle Ax, Ae_j\rangle}{f(x)}.$$

Application of Corollary 2.4.3.(iii) and (ii) as well as part (ii) implies

$$D_j^2 f(x) = D(D_j f)(x)e_j = \frac{\|Ae_j\|^2}{f(x)} - \frac{\langle Ax, Ae_j\rangle^2}{f^3(x)}.$$

(iv) Summation of the preceeding identities for $j$ running from 1 to $n$ gives

$$\triangle f(x) = \sum_{1\le j\le n} D_j^2 f(x) = \frac{1}{f(x)} \sum_{1\le j\le n} \|Ae_j\|^2 - \frac{1}{f^3(x)} \sum_{1\le j\le n} \langle A^t Ax, e_j\rangle^2.$$

Furthermore, note that, for all $y \in \mathbb{R}^n$,

$$\sum_{1\le j\le n} \langle y, e_j\rangle^2 = \Big\| \sum_{1\le j\le n} \langle y, e_j\rangle e_j \Big\|^2 = \|y\|^2.$$

(v) In this case we obtain $\triangle(\|\cdot\|)(x) = \frac{n-1}{\|x\|}$, for $x \in \mathbb{R}^n\backslash\{0\}$.

**Exercise 0.3**

(i) We have $D\phi(x) = \begin{pmatrix} 3(\frac{x_1}{x_2})^2 & -2(\frac{x_1}{x_2})^3 \\ -2(\frac{x_2}{x_1})^3 & 3(\frac{x_2}{x_1})^2 \end{pmatrix}$ and so $detD\Phi(x) = 9 - 4 = 5$.

(ii) Given arbitrary $y \in \mathbb{R}^2_+$, consider the equation $\Phi(x) = y$ for $x \in \mathbb{R}^2_+$; then $\frac{x_1^3}{x_2^2} = y_1$ and $\frac{x_2^3}{x_1^2} = y_2$. Multiplication and division of these equalities leads to

$$x_1x_2 = y_1y_2 \quad \text{and} \quad (\frac{x_1}{x_2})^5 = \frac{y_1}{y_2}. \quad \text{So} \quad x_1x_2 = y_1y_2 \quad \text{and} \quad \frac{x_1}{x_2} = \frac{y_1^{\frac{1}{5}}}{y_2^{\frac{1}{5}}},$$

and multiplication of the equalities now gives $x_1^2 = y_1^{\frac{6}{5}} y_2^{\frac{4}{5}}$. Accordingly, $x_1 = y_1^{\frac{3}{5}} y_2^{\frac{2}{5}} = (y_1y_2)^{\frac{2}{5}} y_1^{\frac{1}{5}}$ because $x_1$, $y_1$ and $y_2 \in \mathbb{R}_+$. Similarly, we obtain the desired formula for $x_2$. It follows that $\Phi$ and $\Psi$ are each other's inverses. On $\mathbb{R}^2_+$ the mapping $\Psi$ is of class $C^\infty$, which implies that $\Phi$ is a $C^\infty$ diffeomorphism. From part (i) and the multiplicative property of the determinant we obtain $detD\Phi(y) = \frac{1}{5}$.

[1] Deze uitwerkingen zijn met de grootste zorg gemaakt. In geval van fouten kan de $\mathcal{T}_{\!\mathcal{B}}\mathcal{C}$ niet verantwoordelijk worden gesteld, maar wordt zij wel graag op de hoogte gesteld: tbc@a-eskwadraat.nl

(iii) We have

$$g \circ \Psi(y) = (y_1y_2)^2(y_1 + y_2) - 5a(y_1y_2)^{\frac{8}{5}}(y_1y_2)^{\frac{2}{5}} = (y_1y_2)^2(y_1 + y_2 - 5a).$$

This implies $x = \Psi(y) \in U$ if and only if $g(x) = (y_1y_2)^2(y_1+y_2-5a) < 0$ if and only $y_1+y_2-5a$ if and only if $y \in V$.

**Exercise 0.4**

(i) We have

$$Dg(x) = 5(x_1(x_1^3 - 2ax_2^2), x_2(x_2^3 - 2ax_1^2)).$$

This matrix is of rank 1 unless (a) $x = 0$ or (b) $x_1^3 = 2ax_2^2$ and $x_2^3 = 2ax_1^2$. In case (b) we may assume $x \neq 0$ and we also obtain $x_1^9 = 8a^3x_2^6 = 32a^5x_1^4$, that is, $x_1^5 = (2a)^5$, which holds if and only if $x_1 = 2a$. In turn this implies $x_2 = 2a$, but $g(2a, 2a) = 64a^5 - 80a^5 = -16a^5 < 0$; in other words, $(2a, 2a) \notin F$. It follows that $g$ is submersive at every point of $F\backslash\{0\}$. The desired conclusion follows from the Submersion Theorem 4.5.2.(ii).

(ii) We eliminate $x_2$ from the equations $g(x) = 0$ and $x_2 = tx_1$, for fixed $t \in \mathbb{R}$. This leads to $(1 + t^5)x_1^5 = 5at^2x_1^4$, with solutions $x_1 = 0$ (as was to be expected) or $x_1 = \frac{5at^2}{1+t^5}$; and the desired formula for $\phi$ holds.

(iii) The formula for $\phi'$ is a consequence of

$$\phi'(t) = \frac{5a}{(1+t^5)^2}\begin{pmatrix} 2t(1+t^5) - t^2 5t^4 \\ 3t^2(1+t^5) - t^3 5t^4 \end{pmatrix} = \frac{5at}{(1+t^5)^2}\begin{pmatrix} 2 + 2t^5 - 5t^5 \\ t(3 + 3t^5 - 5t^5) \end{pmatrix}.$$

If $t \neq 0$, then the assumption $\phi'(t) = 0$ implies $2 - 3t^5 = 0$ and $3 - 2t^5 = 0$. This gives $9t^5 = 6 = 4t^5$, that is $5t^5 = 0$, and so arrived at a contradiction. Therefore $\phi'(t) \neq 0$ if $t \neq 0$; hence $\phi'(t)$ is of rank 1, which proves that $\phi$ is everywhere immersive except at 0.

(iv) $F$ has self-intersection at 0 as follows from $\lim_{t\to\pm\infty} \phi(t) = 0 = \phi(0)$. Indeed, $\widetilde{\phi}: \mathbb{R}\backslash\{-1\} \to \mathbb{R}^2$ with $\widetilde{\phi}(u) = \phi(\frac{1}{u})$ also defines a parametrization of $F$. Now $\phi(t)$ approaches 0 in a vertical direction as $t \downarrow 0$, while $\widetilde{\phi}(u)$ approaches 0 in a horizontal direction as $u \downarrow 0$.

(v) Select $t_0 > 0$ sufficiently small, that is, suppose $2-3t_0^5 > 0$ and $3-2t_0^5 > 0$. For $t$ running from -1 to $t_0$, the sign of the first component $t(2 - 3t^5)$ of $\phi'(t)$ changes from negative to positive at $t = 0$, whereas the sign of the second component $t^2(3 - 2t^5)$ remains nonnegative and vanishes for $t = 0$ only. This behavior of $\phi'$ near 0 is characteristic for a vertical cusp.