TWEEDE DEELTENTAMEN WISB 212

Analyse in Meer Variabelen

03–07–2007 14–17 uur

- *Zet uw naam en collegekaartnummer op elk blad alsmede het totaal aantal ingeleverde bladzijden.*
- *De verschillende onderdelen van het vraagstuk zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*
- *Bij dit tentamen mogen boeken, syllabi, aantekeningen en/of rekenmachine* **NIET** *worden gebruikt.*
- *De twee vraagstukken tellen ieder voor de helft van het totaalcijfer.*
- *Het tentamen telt* **VIER** *bladzijden.*

Exercise 0.1 (Adjoints, vector calculus and quaternions). Write C for the linear space $C_c^\infty(\mathbf{R}^3)$ of C^∞ functions on $\mathbf{R}^3$ with compact support and introduce the usual inner product on C by $\langle\, f, g\,\rangle_C = \int_{\mathbf{R}^3} f(x)g(x)\,dx$, for f and $g \in C$. Consider the linear operator $D_j : C \to C$ of partial differentiation with respect to the j-th variable, for $1 \leq j \leq 3$.

(i) Prove that D_j is anti-adjoint with respect to the inner product on C, that is,

$$\langle\, D_j f,\, g\,\rangle_C = -\langle\, f,\, D_j g\,\rangle_C.$$

Denote by V the linear space of C^∞ vector fields on $\mathbf{R}^3$ with compact support and introduce an inner product on V by $\langle\, v,\, w\,\rangle_V = \int_{\mathbf{R}^3}\langle\, v(x),\, w(x)\,\rangle\,dx$, for v and $w \in V$. Here the inner product at the right-hand side is the usual inner product of vectors in $\mathbf{R}^3$. Furthermore, consider the linear operators $\mathrm{grad} : C \to V$ and $\mathrm{div} : V \to C$.

(ii) For $f \in C$ and $v \in V$, verify the following identity of functions in C:

$$\mathrm{div}(f\,v) = \langle\,\mathrm{grad}\,f,\, v\,\rangle + f\,\mathrm{div}\,v.$$

Use this to prove

$$\langle\,\mathrm{grad}\,f,\, v\,\rangle_V = -\langle\, f,\, \mathrm{div}\,v\,\rangle_C.$$

Conclude that $-\,\mathrm{div} : V \to C$ is the adjoint operator of $\mathrm{grad} : C \to V$.

(iii) For v and w in V, prove the following identity of functions in C:

$$\mathrm{div}(v \times w) = \langle\,\mathrm{curl}\,v,\, w\,\rangle - \langle\, v,\, \mathrm{curl}\,w\,\rangle.$$

Hint: At the left-hand side the operator D_1 only occurs in the term $D_1(v_2w_3 - v_3w_2)$ and apply Leibniz' rule. Next determine the occurrence of D_1 at the right-hand side.

(iv) Deduce from part (iii) that

$$\langle\,\mathrm{curl}\,v,\, w\,\rangle_V = \langle\, v,\, \mathrm{curl}\,w\,\rangle_V.$$

In other words, the linear operator $\mathrm{curl} : V \to V$ is self-adjoint.

Now consider the following matrix of differentiations acting on mappings $\begin{pmatrix} v \\ f \end{pmatrix} : \mathbf{R}^3 \to \mathbf{R}^4$:

$$M = \begin{pmatrix} \mathrm{curl} & \mathrm{grad} \\ -\,\mathrm{div} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -D_3 & D_2 & D_1 \\ D_3 & 0 & -D_1 & D_2 \\ -D_2 & D_1 & 0 & D_3 \\ -D_1 & -D_2 & -D_3 & 0 \end{pmatrix}.$$

The preceding results (in particular, part (i)) imply that M is a symmetric matrix, which **in this context** must be phrased as $M^t = -M$ (when "truly" transposing the matrix we also have to take the transpose of its coefficients).

(v) Verify that $-M^2$ equals Gram's matrix associated to M, that is, the matrix containing the inner products of the column vectors of M. Deduce $M^2 = -\Delta E$, where Δ is the Laplacian and E the 4×4 identity matrix. Derive, for $f \in C$ and $v \in V$

$$\mathrm{curl}\,\mathrm{grad}\,f = 0, \qquad \mathrm{div}\,\mathrm{curl}\,v = 0, \qquad \mathrm{curl}(\mathrm{curl}\,v) = \mathrm{grad}(\mathrm{div}\,v) - \Delta v,$$

where in the third identity the Laplacian Δ acts by components on v. Finally, show how to derive the second identity from the first.

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Background. We may write $M = D_1 I + D_2 J + D_3 K$, where I, J and $K \in \mathrm{Mat}(4, \mathbf{R})$ satisfy $I^2 = J^2 = K^2 = IJK = -E$. As a consequence $IJ = -JI = K$. Phrased differently, the linear space over $\mathbf{R}$ spanned by E, I, J, K provided with these rules of multiplication forms the noncommutative field $\mathbf{H}$ of the *quaternions*. In addition, analogously to the situation in dimension 1 where $(i\frac{d}{dx})^2 = -\frac{d^2}{dx^2}$, we have decomposed the Laplacian on $\mathbf{R}^3$ in a product of matrix-valued linear factors:

$$\Big(\frac{\partial}{\partial x_1}I + \frac{\partial}{\partial x_2}J + \frac{\partial}{\partial x_3}K\Big)^2 = -\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)E.$$

Exercise 0.2 (Left-invariant integration on $\mathrm{Mat}(n, \mathbf{R})$**).** As usual, we write $C_0(\mathbf{R}^n)$ for the linear space of continuous functions $f : \mathbf{R}^n \to \mathbf{R}$ having bounded support. Furthermore, we identify the linear space $\mathrm{Mat}(n, \mathbf{R})$ of $n \times n$ matrices over $\mathbf{R}$ with $\mathbf{R}^{n^2}$; in this way, by using n^2-dimensional integration, we assign a meaning to

$$\int_{\mathrm{Mat}(n,\mathbf{R})} f(X)\, dX \qquad \big(f \in C_0(\mathrm{Mat}(n, \mathbf{R}))\big).$$

(i) In particular, suppose $n = 2$ and consider the subgroup

$$\mathbf{SO}(2, \mathbf{R}) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{Mat}(2, \mathbf{R}) \,\Big|\, -\pi < \alpha \le \pi \right\}$$

of all orthogonal matrices in $\mathrm{Mat}(2, \mathbf{R})$ of determinant 1. Without proof one may use that ϕ is a C^∞ embedding if we define

$$\phi : \,]-\pi, \pi[\, \to \mathbf{R}^4 \qquad \text{by} \qquad \phi(\alpha) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha).$$

Now prove $\mathrm{vol}_1(\mathbf{SO}(2, \mathbf{R})) = 2\pi\sqrt{2}$.

(ii) Prove, for any $f \in C_0(\mathbf{R})$ with $0 \notin \mathrm{supp}\, f$ and any $0 \ne y \in \mathbf{R}$,

$$\int_{\mathbf{R}} \frac{f(y\,x)}{x}\, dx = \int_{\mathbf{R}} \frac{f(x)}{x}\, dx.$$

We now generalize the identity in part (ii) to $\mathrm{Mat}(n, \mathbf{R})$. We shall prove, for every $f \in C_0(\mathrm{Mat}(n, \mathbf{R}))$ with $\mathrm{supp}\, f \subset \mathbf{GL}(n, \mathbf{R})$ (= the group of invertible matrices in $\mathrm{Mat}(n, \mathbf{R})$) and $Y \in \mathbf{GL}(n, \mathbf{R})$,

$$(\star) \qquad \int_{\mathrm{Mat}(n,\mathbf{R})} \frac{f(YX)}{|\det X|^n}\, dX = \int_{\mathrm{Mat}(n,\mathbf{R})} \frac{f(X)}{|\det X|^n}\, dX.$$

Given $Y \in \mathbf{GL}(n, \mathbf{R})$, define

$$\Phi_Y : \mathrm{Mat}(n, \mathbf{R}) \to \mathrm{Mat}(n, \mathbf{R}) \qquad \text{by} \qquad \Phi_Y(X) = Y\,X.$$

(iii) Show that Φ_Y is a C^∞ diffeomorphism satisfying $D\Phi_Y(X) = \Phi_Y$, for all $X \in \mathrm{Mat}(n, \mathbf{R})$.

Denote by $e_1, \dots, e_n$ the standard basis (column) vectors in $\mathbf{R}^n$, then a basis for $\mathrm{Mat}(n, \mathbf{R})$ is formed by the matrices

$$E_{i,j} = (0 \cdots 0\ e_i\ 0 \cdots 0) \qquad (1 \le i,\, j \le n),$$

where e_i occurs in the j-th column. The ordering is lexicographic, but first with respect to j and then to i. In the case of $n = 2$ we thus obtain, in the following order:

$$E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iv) Verify $\Phi_Y(E_{i,j}) = (0 \dots 0\ Y e_i\ 0 \dots 0)$. Deduce that the matrix of Φ_Y with respect to the $(E_{i,j})$ is given in block diagonal form with a copy of Y in each block and that $\det \Phi_Y = (\det Y)^n$.
Hint: First consider explicitly the case of $n = 2$, where the matrix of Φ_Y belongs to $\mathrm{Mat}(4, \mathbf{R})$. Then treat the general case.

(v) Prove $\Phi_Y(\mathbf{GL}(n, \mathbf{R})) \subset \mathbf{GL}(n, \mathbf{R})$. Now show the validity of $(\star)$ above by applying parts (iii) and (iv).

(vi) Select $Y \in \mathbf{GL}(n, \mathbf{R})$ satisfying $\det Y = -1$ and set $f(X) = \det X$. With these data $(\star)$ implies $-1 = 1$. Explain!

Solution of Exercise 0.1

(i) Because f and g are of compact support, it is possible to select an open ball $\Omega \subset \mathbf{R}^n$ containing $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$; in particular, f and g vanish along $\partial\Omega$. The formula then follows from Corollary 7.6.2 because the integral over $\partial\Omega$ vanishes.

(ii) On account of Leibniz' rule we have

$$\mathrm{div}(f\,v) = \sum_{1\le j\le 3} D_j(f\,v_j) = \sum_{1\le j\le 3} (D_j f)\,v_j + \sum_{1\le j\le 3} f\,D_j v_j = \langle\, \mathrm{grad}\, f,\, v\,\rangle + f\,\mathrm{div}\, v.$$

Next integrate this identity over $\mathbf{R}^3$ and notice that Gauss' Divergence Theorem 7.8.5 implies that the integral of the left-hand side equals $\int_{\partial\Omega} f(y)\langle\, v(y),\, \nu(y)\,\rangle\, dy = 0$, for the same reasons as in part (i). The final conclusion is a consequence of the definition of the adjoint in Section 2.1.

(iii) At the left-hand side D_1 occurs in the term $v_2 D_1 w_3 + w_3 D_1 v_2 - v_3 D_1 w_2 - w_2 D_1 v_3$, while at the right-hand side it occurs in $-w_2 D_1 v_3 + w_3 D_1 v_2 + v_2 D_1 w_3 - v_3 D_1 w_2$, which is a rearrangement of the former expression. Taking the indices modulo 3 one obtains analogous results for D_2 and D_3 by means of cyclic permutation of the indices.

(iv) The desired results follow in the same manner as in part (ii).

(v) First note that $-M^2 = M^t M$ where the right-hand side is Gram's matrix according to Section 2.1. On the basis of the symmetry of Gram's matrix and $D_i D_j = D_j D_i$, one has to perform 10 trivial mental calculations to establish that $\langle\, M_i,\, M_j\,\rangle = \delta_{ij}\,\Delta$, for $1 \le i, j \le 3$. This leads to $M^2 = -\Delta E$. One finds on the one hand

$$M^2 = \begin{pmatrix} \mathrm{curl} & \mathrm{grad} \\ -\,\mathrm{div} & 0 \end{pmatrix}\begin{pmatrix} \mathrm{curl} & \mathrm{grad} \\ -\,\mathrm{div} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{curl}\circ\mathrm{curl} - \mathrm{grad}\circ\mathrm{div} & \mathrm{curl}\circ\mathrm{grad} \\ -\,\mathrm{div}\circ\mathrm{curl} & -\,\mathrm{div}\circ\mathrm{grad} \end{pmatrix},$$

while on the other hand it equals $(-\Delta)E$. Comparison of the matrix coefficients leads to the desired conclusions. Observe that in addition one recovers the definition $\Delta = \mathrm{div}\circ\mathrm{grad}$. The second identity follows from the first by taking the transpose.

Solution of Exercise 0.2

(i) We have

$$\|D\phi(\alpha)\| = \|(-\sin\alpha, \cos\alpha, -\cos\alpha, -\sin\alpha)\| = \sqrt{2}.$$

Therefore integration of the constant function 1 over the submanifold $\mathbf{SO}(2,\mathbf{R})$ with respect to the Euclidean density gives $\int_{-\pi}^{\pi} \sqrt{2}\, d\alpha = 2\pi\sqrt{2}$.

(ii) The formula is a direct consequence of the substitution $x \mapsto y\,x$ in the right-hand side of the given formula.

(iii) The coefficients of the product matrix $Y\,X$ are given by polynomial functions in the coefficients of Y and X, therefore Φ_Y is a C^∞ mapping. As $Y \in \mathbf{GL}(n,\mathbf{R})$, the mapping Φ_Y is invertible, with $\Phi_{Y^{-1}}$ as its inverse; and this shows that Φ_Y is a C^∞ diffeomorphism. The formula for $D\Phi_Y$ follows from Example 2.2.5, because Φ_Y is a linear mapping.

(iv) On account of the properties of matrix multiplication we have

$$\begin{aligned} \Phi_Y(E_{i,j}) &= Y\,E_{i,j} = Y\,(0\,\cdots\,0\;e_i\;0\,\cdots\,0) = (Y0\,\cdots\,Y0\;Ye_i\;Y0\,\cdots\,Y0) \\ &= (0\,\cdots\,0\;Ye_i\;0\,\cdots\,0). \end{aligned}$$

The matrix of Φ_Y is obtained by successively applying Φ_Y to all the basis vectors in $\mathrm{Mat}(n, \mathbf{R})$. Since the resulting $n^2 \times n^2$ matrix contains n identical blocks along the diagonal, the formula for $\det \Phi_Y$ follows.

(v) The inclusion is a consequence of the multiplicative property of the determinant. Application of the Change of Variables Theorem 6.6.1 with $\Psi = \Phi_Y$ leads to $(*)$, because $|\det D\Phi_Y(X)| = |\det \Phi_Y| = |\det Y|^n$, for all $X \in \mathrm{Mat}(n, \mathbf{R})$.

(vi) In this case, the function f has no bounded support. Actually, the integral on the right-hand side of $(*)$ is divergent.