Mock exam 2 – Group theory

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) Let D_n be the dihedral group given by

$$D_n = \langle a, b : a^n = b^2 = e; bab^{-1} = a^{-1} \rangle.$$

a) Compute Z_{D_n}, the center of D_n, for $n > 1$. Analyse carefully the cases $n = 2$, n even and greater than 2 and n odd.

b) Show that if $n > 1$, then $D_{2n}/Z_{D_{2n}}$ is isomorphic to D_n.

2) For each list of groups a) and b) below, decide which of the groups within that list are isomorphic, if any:

a) D_3, S_3 and the group generated by

$$\langle a, b : a^3 = b^2 = e; aba^{-1} = ba \rangle.$$

b) D_{12}, $\mathbb{Z}_4 \times D_3$ and S_4.

3) Let G be a finite group. We define a sequence (G_i) of subgroups of G as follows. We let $G_0 = G$ and define inductively G_i as the group generated by

$$G_i = \langle ghg^{-1}h^{-1} : g \in G \text{ and } h \in G_{i-1} \rangle$$

So, for example, G_1 is the commutator subgroup of G.

a) Show that each G_i is subgroup of G_{i-1}. Further, show that $G_i \lhd G_{i-1}$ and that the quotient G_{i-1}/G_i is Abelian.

b) Show that if, for some i_0, $G_{i_0} = G_{i_0+1}$ then $G_n = G_{i_0}$ for all $n > i_0$.

c) Compute the sequence of subgroups G_i above for $G = D_8$, D_{10} and A_5.

4) Show that if G has order $p_1 p_2 \cdots p_n$, for p_i primes with $p_i \leq p_{i+1}$ and $H < G$ is a subgroup of order $p_2 \cdots p_n$, then H is normal.

5) Let G be a group of order np^k, with $n > 1$, $k > 0$, $p > 2$ and n and p coprimes.

a) Show that if $n < p$ then G is not simple,

b) Show that if $n < 2p$ and $k > 1$, then G is not simple,

c) Show that if $k > n/p$ and $n < p^2$, then G is not simple.

6) In what follows let G be a finite group and $K, H < G$. Prove or give counter-examples to the following claims.

a) If $K \lhd G$, then $K \cap H \lhd H$.

b) If K is a p-Sylow of G then $K \cap H$ is a p-Sylow of H.

7) Let $p > 2$. What is the order of a p-Sylow of S_{2p}? Give an example of one such group. Finally, find all p-Sylows of S_{2p}.