Group theory – Exam

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) For each list of groups a) and b) below, decide which of the groups within each list are isomorphic, if any:

a) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_9 \times \mathbb{Z}_2$, $\mathbb{Z}_{18}$ and $\mathbb{Z}_6 \times \mathbb{Z}_3$ (0.5 pt).

b) S_4, $A_4 \times \mathbb{Z}_2$, D_{12} and $\mathbb{H} \times \mathbb{Z}_3$, where $\mathbb{H}$ is the quaternion group with 8 elements (0.5 pt).

2) Show that if a finite group G has only two conjugacy classes, then $G \cong \mathbb{Z}_2$ (1.0 pt).

3 a) Show that if S_n acts on a set with p elements and $p > n$ is a prime number then the action has more than one orbit (0.75 pt).
b) Let p be a prime. Show that the only action of $\mathbb{Z}_p$ on a set with $n < p$ elements is the trivial one (0.75 pt).

4) Prove or give a counter-example for the following claim: For every m which divides 60 there is a subgroup of A_5 of order m (1.5 pt).

5) Let G be a finite group. We define a sequence of groups (G_i) as follows. Let $G_0 = G$ and define inductively $G_i = G_{i-1}/Z_{G_{i-1}}$, where $Z_{G_{i-1}}$ is the center of G_{i-1}, so for example, $G_1 = G/Z_G$. This procedure gives rise to a sequence of groups

$$G = G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

where each map $G_{i-1} \longrightarrow G_i$ is a surjective group homomorphism whose kernel is the center of G_{i-1}.

a) Show that if $Z_{G_i} = \{e\}$ for some i, then $G_n = G_i$ for $n > i$ (0.3 pt).

b) Show that if G_i is Abelian, then $G_n = \{e\}$ for $n > i$ (0.3 pt).

c) Compute this sequence for D_8, D_{10} and A_5 (0.9 pt).

6) Prove or give a counter example to the following claim: Let G_1 and G_2 be finite groups and $H_1 \lhd G_1$, $H_2 \lhd G_2$ be normal subgroups such that $H_1 \cong H_2$. If $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$ (1.5 pt).

7) Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Show that the 11 and the 7-Sylows are normal. Show that the 11-Sylow is in the center of G (1.5 pt).

8) Show that a group of order $392 = 2^3 \cdot 7^2$ is not simple (1.5 pt).