Group theory – Exam

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) For each list of groups a) and b) below, decide which of the groups within each list are isomorphic, if any:

a) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_9 \times \mathbb{Z}_2$, $\mathbb{Z}_{18}$ and $\mathbb{Z}_6 \times \mathbb{Z}_3$ (0.5 pt).

b) S_4, $A_4 \times \mathbb{Z}_2$, D_{12} and $\mathbb{H} \times \mathbb{Z}_3$, where $\mathbb{H}$ is the quaternion group with 8 elements (0.5 pt).

Solution. a) For this one you should remember that $\mathbb{Z}_n \times \mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{nm}$ if and only if n and m are coprime. This implies that $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ and $\mathbb{Z}_9 \times \mathbb{Z}_2 \cong Z_{18}$, but that $\mathbb{Z}_3 \times \mathbb{Z}_6 \not\cong \mathbb{Z}_{18}$.
b) First consider the center of each of the groups given. We have $Z_{S_4} = \{e\}$, $Z_{A_4 \times \mathbb{Z}_2} \cong Z_{A_4} \times Z_{\mathbb{Z}_2} \cong \mathbb{Z}_2$, $Z_{D_12} \cong \mathbb{Z}_2$ and $Z_{H \times \mathbb{Z}_3} \cong Z_H \times Z_{\mathbb{Z}_3} = \mathbb{Z}_2 \times \mathbb{Z}_3$. So the only two groups which can be isomorphic are D_{12} and A_4, but D_{12} has an element of order 6 and A_4 does not, so no two groups of list b) are isomorphic.
Compare this exercise with exercise 10.7 in Armstrong (exercise sheet 7).

□

2) Show that if a finite group G has only two conjugacy classes, then $G \cong \mathbb{Z}_2$ (1.0 pt).

Solution. Let G be such a group and let n be its order. The identity element e is always in a conjugacy class of its own and every element is in a conjugacy class, so, since G has only two conjugacy classes, we have that $G\backslash\{e\}$ must be a conjugacy class. Since a conjugacy class is an orbit of the action of G on itself by conjugation, the orbit stabilizer theorem says that the order of an orbit must divide the order of the group, i.e.,

$$n - 1 = \#(G\backslash\{e\}) | \#G = n,$$

this can only happen if $n = 1$ or $n = 2$. For $n = 1$ G is the trivial group which has only one conjugacy class, so, since G has two classes, $\#G = 2$ and hence $G \cong \mathbb{Z}_2$.
This was exercise 6 from hand-in sheet 3.

□

3 a) Show that if S_n acts on a set with p elements and $p > n$ is a prime number then the action has more than one orbit (0.75 pt).

b) Let p be a prime. Show that the only action of $\mathbb{Z}_p$ on a set with $n < p$ elements is the trivial one (0.75 pt).

Solution. a) Assume by contradiction that the action has only one orbit. Then, by the orbit stabilizer theorem, the cardinality of the set (i.e. the only orbit) must divide the order of the group acting on it. Therefore, for the present case, p must divide $\#S_n = n!$. Since p is a prime, it divides a product if and only if it divides one of the factors, but since $n < p$, p does not divide any factor in $n!$. This contradiction implies that there is more than one orbit.
b) Let X be a set with n elements on which $\mathbb{Z}_p$ acts and let $x \in X$. By the orbit stabilizer theorem, $p = \mathbb{Z}_p$ must divide the order of the orbit through x, $\mathcal{O}_x$, so $\#\mathcal{O}_x = 1$ or $\mathcal{O}_x = p$. Since $\mathcal{O}_x \subset X$, $\#\mathcal{O}_x \leq n < p$, hence $\#\mathcal{O}_x = 1$. Since x is an arbitrary element of X, this means that the action is trivial.
Exercise a)was exercise 5 from hand-in sheet 3 and also exercise 5 from sheet 6.

□

4) Prove or give a counter-example for the following claim: For every m which divides 60 there is a subgroup of A_5 of order m (1.5 pt).

Solution. Recall that A_5 is simple, i.e., it has no nontrivial normal subgroups. Since every subgroup of index 2 is normal, this means that A_5 has no subgroup of order 30.
This was exercise 11.8 from Armstrong (exercise sheet 7).

□

5) Let G be a finite group. We define a sequence of groups (G_i) as follows. Let $G_0 = G$ and define inductively $G_i = G_{i-1}/Z_{G_{i-1}}$, where $Z_{G_{i-1}}$ is the center of G_{i-1}, so for example, $G_1 = G/Z_G$. This procedure gives rise to a sequence of groups

$$G = G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

where each map $G_{i-1} \longrightarrow G_i$ is a surjective group homomorphism whose kernel is the center of G_{i-1}.

a) Show that if $Z_{G_i} = \{e\}$ for some i, then $G_n = G_i$ for $n > i$ (0.3 pt).

b) Show that if G_i is Abelian, then $G_n = \{e\}$ for $n > i$ (0.3 pt).

c) Compute this sequence for D_8, D_{10} and A_5 (0.9 pt).

Solution. a) We prove this by induction. If $Z_{G_i} = \{e\}$, then $G_{i+1} = G_i/Z_{G_i} \cong G_i/\{e\} \cong G_i$, showing the first step holds. The inductive hypothesis is that $G_j \cong G_i$, and then $Z_{G_j} \cong Z_{G_i} = \{e\}$ and $G_{j+1} = G_j/Z_{G_j} \cong G_j/\{e\} \cong G_j \cong G_i$, proving the inductive step.
b) If G_i is Abelian, then $Z_{G_i} = G_i$ and $G_{i+1} = G_i/Z_{G_i} = G_i/G_i \cong \{e\}$. So, $Z_{G_{i+1}} = \{e\}$ and by part a) $G_j \cong G_{i+1} \cong \{e\}$ for $j > i$.
c) D_{10} and D_8. We know that for $n > 1$, $Z_{D_{2n-1}} \cong \{e\}$ and $Z_{D_{2n}} \cong \mathbb{Z}_2$ and that $D_{2n}/Z_{D_{2n}} \cong D_n$. We also know that $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is Abelian, hence $Z_{D_2} = D_2$. These facts give us.

For $G = D_{10}$, we have $G_0 = D_{10}$, $G_1 = D_5$ and $Z_{D_5} = \{e\}$, so, by a), $G_i = D_5$ for $i > 0$.

For $G = D_8$, we have $G_0 = D_8$, $G_1 = D_4$, $G_2 = D_2$ and $G_i = \{e\}$ for $i > 2$.

Finally, A_5 has trivial center, since it is simple and is not Abelian (and the center is always a normal subgroup). So, by a), $G_i = A_5$ for all i.
Compare this with exercise 2 of hand-in sheet 5.

□

6) Prove or give a counter example to the following claim: Let G_1 and G_2 be finite groups and $H_1 \lhd G_1$, $H_2 \lhd G_2$ be normal subgroups such that $H_1 \cong H_2$. If $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$ (1.5 pt).

Solution. False: Consider $G_1 = \mathbb{Z}_4$ and $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. By the same argument of the first exercise of this exam, G_1 and G_2 are not isomorphic. However, $H = Z_2$ is a subgroup of both of these groups and since G_1 and G_2 are Abelian, any subgroup is normal. Further, G_i/H is a group with 2 elements, hence isomorphic to $\mathbb{Z}_2$. This gives a counter-example to the claim.

$\square$

7) Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Show that the 11 and the 7-Sylows are normal. Show that the 11-Sylow is in the center of G (1.5 pt).

Solution. By the Sylow theorems, n_{11}, the number of 11-Sylow subgroups, must divide $\#G/11 = 3 \cdot 7$, i.e., it can be only 1, 3, 7 or 11. Further it must be 1 mod 11, so there is only 1 11-Sylow which therefore is a normal subgroup.

By the Sylow theorems, n_7, the number of 7-Sylow subgroups, must divide $\#G/7 = 3 \cdot 11$, i.e., it can be only 1, 3, 11 or 33. Further it must be 1 mod 7, so there is only 1 7-Sylow which therefore is a normal subgroup.

Let H be the 11- Sylow,so that $H \cong \mathbb{Z}_{11}$ and let $g \in G$. Then, by Lagrange, g must have order $1, 3, 7, 11, 21, 33, 77$ or $231 = 3 \cdot 7 \cdot 11$. We will consider the case when g has order 21 as the other cases are similar. If g has order 21 and x is a generator of H, then, by normality of H we have

$$gxg^{-1} = x^l$$

for some l. Therefore

$$x = g^{21}xg^{-21} = x^{l^{21}},$$

showing that $l^{21} = 1 \mod 11$. But $\mathbb{Z}_{11}\backslash\{0\}$ is a group (with multiplication) and the order of any of its elements must divide $10 = \#\mathbb{Z}_{11}\backslash\{0\}$. From $l^{21} = 1 \mod 11$, we see that the order of l must divide 21. Since 21 and 10 are coprime, the only number which divides both is 1, so l has order 1 and hence it is the identity of $\mathbb{Z}_{11}\backslash\{0\}$, showing that $gxg^{-1} = x$ and hence g commutes with x.
This is exercise 4 from the last exercise sheet and is also a particular case of exercise 1 of hand-in sheet 6.

$\square$

8) Show that a group of order $392 = 2^3 \cdot 7^2$ is not simple (1.5 pt).

Solution. One way to solve:
By Sylow's theorem, n_7, the number of 7-Sylows must divide 8 and be 1 mod 7, so there are either 1 or 8 7-Sylows. If there is only 1, then the 7-Sylow is normal and hence G is not simple. If $n_7 = 8$, then let $\mathcal{H}$ be the set whose elements are the 7-Sylows of G. Then G acts of $\mathcal{H}$ by conjugation:

$$G \times \mathcal{H} \longrightarrow \mathcal{H} \qquad g : H \mapsto gHg^{-1} \in \mathcal{H}$$

Since $\mathcal{H}$ has 8 elements, this is equivalent to a group homomorphism

$$\varphi : G \longrightarrow S_8.$$

By Sylow's theorem, any two 7-Sylow subgroups are conjugated to each other, hence the action above is not trivial, so $\ker(\varphi) \neq G$. By the first isomorphism theorem and Lagrange we have that

$$\#G = \#\mathrm{Im}(\varphi)\#\ker\varphi$$

and since $\mathrm{Im}(\varphi) < S_8$ we see that $\#\mathrm{Im}(\varphi)|8!$. Since $7^2 \not| 8!$, we have that $7^2 \not| \#\mathrm{Im}(\varphi)$, but $7^2|\#G$, so $\#\ker(\varphi) \neq 1$ and therefore $\ker(\varphi) \neq \{e\}$.

The kernel of any homomorphism is a normal subgroup and $\ker(\varphi) \neq G, \{e\}$, so it is a nontrivial normal subgroup of G, hence G is not simple.

Another way to solve (very similar argument):

Let H be a 7-Sylow subgroup of G. Then H has index 8, i.e., the set $\mathcal{H}$ of left H-cosets has 8 elements. Then G acts of $\mathcal{H}$ by left translation:

$$G \times \mathcal{H} \longrightarrow \mathcal{H} \qquad g : xH \mapsto gxH \in \mathcal{H}$$

Since $\mathcal{H}$ has 8 elements, this is equivalent to a group homomorphism

$$\varphi : G \longrightarrow S_8.$$

By Sylow's theorem, any two 7-Sylow subgroups are conjugated to each other, hence the action above is not trivial, so $\ker(\varphi) \neq G$. By the first isomorphism theorem and Lagrange we have that

$$\#G = \#\mathrm{Im}(\varphi)\# \ker \varphi$$

and since $\mathrm{Im}(\varphi) < S_8$ we see that $\#\mathrm{Im}(gf)|8!$. Since $7^2 \nmid 8!$, we have that $7^2 \nmid \#\mathrm{Im}(\varphi)$, but $7^2|\#G$, so $\# \ker(\varphi) \neq 1$ and therefore $\ker(\varphi) \neq \{e\}$.

The kernel of any homomorphism is a normal subgroup and $\ker(\varphi) \neq G, \{e\}$, so it is a nontrivial normal subgroup of G, hence G is not simple.

Compare this to exercise 7 of the last exercise sheet and exercise 3 of hand-in sheet 6.

$\square$

Remark: The first proof can be used in more general conditions than the second, for example, if G had order $2^5 \times 7^2$ the first solution would still hold, but the second would not.