Group theory – Exam

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) For each list of groups a) and b) below, decide which of the groups within each list are isomorphic, if any:

a) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_9 \times \mathbb{Z}_2$, $\mathbb{Z}_{18}$ and $\mathbb{Z}_6 \times \mathbb{Z}_3$ (0.5 pt).

b) S_4, $A_4 \times \mathbb{Z}_2$, D_{12} and $\mathbb{H} \times \mathbb{Z}_3$, where $\mathbb{H}$ is the quaternion group with 8 elements (0.5 pt).

2) Show that if a finite group G has only two conjugacy classes, then $G \cong \mathbb{Z}_2$ (1.0 pt).

3 a) Show that if S_n acts on a set with p elements and $p > n$ is a prime number then the action has more than one orbit (0.75 pt).
b) Let p be a prime. Show that the only action of $\mathbb{Z}_p$ on a set with $n < p$ elements is the trivial one (0.75 pt).

4) Prove or give a counter-example for the following claim: For every m which divides 60 there is a subgroup of A_5 of order m (1.5 pt).

5) Let G be a finite group. We define a sequence of groups (G_i) as follows. Let $G_0 = G$ and define inductively $G_i = G_{i-1}/Z_{G_{i-1}}$, where $Z_{G_{i-1}}$ is the center of G_{i-1}, so for example, $G_1 = G/Z_G$. This procedure gives rise to a sequence of groups

$$G = G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

where each map $G_{i-1} \longrightarrow G_i$ is a surjective group homomorphism whose kernel is the center of G_{i-1}.

a) Show that if $Z_{G_i} = \{e\}$ for some i, then $G_n = G_i$ for $n > i$ (0.3 pt).

b) Show that if G_i is Abelian, then $G_n = \{e\}$ for $n > i$ (0.3 pt).

c) Compute this sequence for D_8, D_{10} and A_5 (0.9 pt).

6) Prove or give a counter example to the following claim: Let G_1 and G_2 be finite groups and $H_1 \lhd G_1$, $H_2 \lhd G_2$ be normal subgroups such that $H_1 \cong H_2$. If $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$ (1.5 pt).

7) Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Show that the 11 and the 7-Sylows are normal. Show that the 11-Sylow is in the center of G (1.5 pt).

8) Show that a group of order $392 = 2^3 \cdot 7^2$ is not simple (1.5 pt).

1) For each list of groups a) and b) below, decide which of the groups within each list are isomorphic, if any:

a) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_6 \times \mathbb{Z}_5$, $\mathbb{Z}_{30}$ and $\mathbb{Z}_2 \times \mathbb{Z}_{15}$ (0.5 pt).

b) S_4, $A_4 \times \mathbb{Z}_2$, D_{12} and $\mathbb{H} \times \mathbb{Z}_3$, where $\mathbb{H}$ is the quaternion group with 8 elements (0.5 pt).

2) Let $\mathcal{S} \subset S_5$ be the set of 5-cycles, sitting inside the group of permutations of 5 elements. Then S_5 acts on $\mathcal{S}$ by conjugation:

$$\sigma \cdot \tau := \sigma\tau\sigma^{-1}, \qquad \sigma \in S_5 \quad \tau \in \mathcal{S}.$$

Compute the orbit and the stabilizer of the 5-cycle (1 2 3 4 5). (1.0 pt).

3) Let G be a finite group and $x \in G$.

a) Show that the set of elements of G which commute with x is a subgroup of G. This subgroup is denoted by $C(x)$. (0.75 pt)

b) Show that the index of $C(x)$ in G is the number of elements in the conjugacy class of x. (0.75 pt)

4 a) Let $n > 4$. Show that if A_n acts on a set with $m < n$ elements then each orbit has size 1. (0.75 pt).

b) Show that if $\mathbb{Z}_p$ acts on a set and p is prime, then each orbit has size 1 or p. (0.75 pt)

5) Let G be a finite group. We define a sequence of groups (G_i) as follows. Let $G_0 = G$ and define inductively $G_i = G_{i-1}/Z_{G_{i-1}}$, where $Z_{G_{i-1}}$ is the center of G_{i-1}, so for example, $G_1 = G/Z_G$. This procedure gives rise to a sequence of groups

$$G = G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

where each map $G_{i-1} \longrightarrow G_i$ is a surjective group homomorphism whose kernel is the center of G_{i-1}.

a) Show that if $Z_{G_i} = \{e\}$ for some i, then $G_n = G_i$ for $n > i$ (0.3 pt).

b) Show that if G_i is Abelian, then $G_n = \{e\}$ for $n > i$ (0.3 pt).

c) Compute this sequence for S_5, D_8 and D_{10} (0.9 pt).

6) Let G be a group of order $385 = 5 \cdot 7 \cdot 11$. Show that the 11 and the 7-Sylows are normal. Show that the 7-Sylow is in the center of G (1.5 pt).

7) Show that a group of order $132 = 2^2 \cdot 3 \cdot 11$ is not simple (1.5 pt).

8 a) Let G act on a set $\mathcal{X}$, let $p \in \mathcal{X}$ and let H be the stabilizer of p. Show that the stabilizer of $g \cdot p$ is the subgroup gHg^{-1}. Conclude that H is normal if and only if it is the stabilizer all the points in the orbit of p. (0.5 pt)

b) Let H be a subgroup of a finite group G and let $\mathcal{X}$ be the set of left H-cosets. Show that the formula

$$g(xH) = gxH$$

defines an action of G on $\mathcal{X}$ and hence it also defines an action of H on $\mathcal{X}$. Prove that H is a normal subgroup of G if and only if every orbit of the induced action of H on $\mathcal{X}$ is trivial, i.e., if and only if

$$hxH = xH \qquad \text{for all } h \in H, x \in G. \text{ (0.5 pt)}$$

c) Let G be a finite group and let p be the smallest prime which divides the order of G. Show that if $H < G$ is a subgroup of index p (i.e., H has exactly p left cosets) then H is normal (hint: use the
1) Let D_n be the dihedral group given by

$$D_n = \langle a, b : a^n = b^2 = e; bab^{-1} = a^{-1} \rangle.$$

a) Compute Z_{D_n}, the center of D_n, for $n > 1$. Analyse carefully the cases $n = 2$, n even and greater than 2 and n odd.

b) Show that if $n > 1$, then $D_{2n}/Z_{D_{2n}}$ is isomorphic to D_n.

2) For each list of groups a) and b) below, decide which of the groups within that list are isomorphic, if any:

a) D_3, S_3 and the group generated by

$$\langle a, b : a^3 = b^2 = e; aba^{-1} = ba \rangle.$$

b) D_{12}, $\mathbb{Z}_4 \times D_3$ and S_4.

3) Let G be a finite group. We define a sequence (G_i) of subgroups of G as follows. We let $G_0 = G$ and define inductively G_i as the group generated by

$$G_i = \langle ghg^{-1}h^{-1} : g \in G \text{ and } h \in G_{i-1} \rangle$$

So, for example, G_1 is the commutator subgroup of G.

a) Show that each G_i is subgroup of G_{i-1}. Further, show that $G_i \triangleleft G_{i-1}$ and that the quotient G_{i-1}/G_i is Abelian.

b) Show that if, for some i_0, $G_{i_0} = G_{i_0+1}$ then $G_n = G_{i_0}$ for all $n > i_0$.

c) Compute the sequence of subgroups G_i above for $G = D_8$, D_{10} and A_5.

4) Show that if G has order $p_1 p_2 \cdots p_n$, for p_i primes with $p_i \leq p_{i+1}$ and $H < G$ is a subgroup of order $p_2 \cdots p_n$, then H is normal.

5) Let G be a group of order np^k, with $n > 1$, $k > 0$, $p > 2$ and n and p coprimes.

a) Show that if $n < p$ then G is not simple,

b) Show that if $n < 2p$ and $k > 1$, then G is not simple,

c) Show that if $k > n/p$ and $n < p^2$, then G is not simple.

6) In what follows let G be a finite group and $K, H < G$. Prove or give counter-examples to the following claims.

a) If $K \triangleleft G$, then $K \cap H \triangleleft H$.

b) If K is a p-Sylow of G then $K \cap H$ is a p-Sylow of H.

7) Let $p > 2$. What is the order of a p-Sylow of S_{2p}? Give an example of one such group. Finally, find all p-Sylows of S_{2p}.

1) For each list of groups a) and b) below, decide which of the groups within each list are isomorphic, if any:

a) $\mathbb{Z}_{20}$, $\mathbb{Z}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$.

b) $\mathbb{Z}_2 \times D_7$, $\mathbb{Z}_2 \times \mathbb{Z}_{14}$, D_{14}.

2) Let G be the set of sequences of integers endowed with the following product operation $+ : G \times G \longrightarrow G$

$$(a_1, a_2, \cdots, a_n, \cdots) + (b_1, b_2, \cdots, b_n, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots).$$

Show that this operation makes G into a group. Show that $\mathbb{Z} \times G \cong G$ and hence conclude that, for groups, it may be the case that $A \times C \cong B \times C$ even though $A \ncong B$.[1]

[1] I'd never ask this in an exam, but at home you may try to prove that *for finite groups* it is true that $A \times C \cong B \times C$ implies $A \cong B$. If you just want to see a proof, take a look at Hirshon's paper On cancellation in groups.

3) Let $n > m$ be natural numbers, $n > 4$, let X be a set with m elements. Show that the orbits of any action of S_n on X have size 1 or 2.

4) Let G be a group, S_G be group of bijections from G into itself and $\mathrm{Aut}(G) \subset S_G$ be the group of automorphisms of G. Consider the map $\mathrm{Ad} : G \longrightarrow S_G$, given by

$$\mathrm{Ad}(g) : G \longrightarrow G \qquad \mathrm{Ad}(g)(x) = gxg^{-1}.$$

a) Show that $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$, i.e., for every $g \in G$, $\mathrm{Ad}(g) : G \longrightarrow G$ is an automorphism;

b) Show that $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$ is a group homomorphism and that the image of Ad is a normal subgroup of $\mathrm{Aut}(G)$. The image of Ad is called the *group of inner automorphisms*.

c) Show that the kernel of $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$ is the center of G and conclude that the group of inner automorphisms is isomorphic to the quotient G/Z_G.

d) Give an example of a group which has an automorphism which is not an inner automorphism.

5) Classify all groups or order $2009 = 7^2 \cdot 41$.

6) Let G be a group and $n \in \mathbb{N}$

a) Let $H_i < G$ be subgroups, for $i \in \{1, \cdots, n\}$, show that

$$\cap_{i=1}^{n} H_i$$

is a subgroup of G.

b) If G is finite and p be a prime. Show that the intersection of all p-Sylows of G is a normal subgroup.

7) Let G be a finite group and $K, H < G$. Prove or give a counter-example to the following claims.

a) If $K \lhd H$ and $H \lhd G$ then $K \lhd G$.

b) If K is the only p-Sylow of G, then $K \cap H$ is a p-Sylow of H.