Group theory – Exam 2

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1a) Is $\mathbb{Z}$ isomorphic to $\mathbb{Q}$? (0.75 pt)
1b) Is $\mathbb{Z}$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$? (0.75 pt)

2) Let G be a finite group which acts transitively on a set X, i.e., the whole X is the orbit of any of its points. Show that if H is a normal subgroup of G then all the orbits of the induced action of H on X have the same size. (1.5 pt)

3) Suppose a group G contains an infinite cyclic subgroup of index 2. Show that G must be isomorphic to the infinite dihedral group

$$D_\infty = \langle a, b | b^2 = e;\ bab^{-1} = a^{-1} \rangle.$$

or to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}_2$. (1.5 pt)

4) Let p and q be primes with $p > q$ and

$$p^2 \neq 1 \bmod q \qquad \text{and} \qquad q^2 \neq 1 \bmod p.$$

Classify all groups of order $p^2 \cdot q^2$. (1.5 pt)

5) Show that a group of order $2^3 \cdot 3 \cdot 23^2$ is not simple. (1.5 pt)

6) Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Show that the 11 and the 7-Sylows are normal. Show that the 11-Sylow is in the center of G (1.5 pt).

7) Show that elements of the unitary group $U(2)$ have the form

$$\begin{pmatrix} z & w \\ -e^{i\theta}\overline{w} & e^{i\theta}\overline{z} \end{pmatrix}$$

where $z, w \in \mathbb{C}$, $\theta \in \mathbb{R}$ and $z\overline{z} + w\overline{w} = 1$. Which of these matrices belong to $SU(2)$. (1.5 pt).

8) Prove or give a counterexample to the following claim: Let G be a finite group, $H < G$ be a subgroup and p be a prime which divides the order of H and G. If K is the only p-Sylow subgroup of G, then $K \cap H$ is the only p-Sylow of H. (1.5 pt)