GROUP THEORY 2013 — SOLUTIONS

EX. 1

ⓐ Since "sign" is multiplicative, a 3-cycle is even,

$\boxed{\text{Sign}(\sigma) = 1}$

ⓑ $(123)(13)(132)(12) = (1)(2)(3) = \boxed{e}$

ⓒ $r^5 s^{11} r^{2013} = r^5 s r^4 = r \cdot \underbrace{r^4 s r^4}_{=s} = rs = \boxed{sr}$

$\uparrow$
$r^7 = e,\ 2013 \equiv 4 (7)$
$s^2 = e$

EX. 2

ⓐ $(\mathbb{Z}/5)^*_\cdot = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\} \cong \mathbb{Z}/4, +$

$(\mathbb{Z}/6)^*_\cdot = \{\bar{1}, \bar{5}\} \cong \mathbb{Z}/2, +$

$(\mathbb{Z}/5)^* \times (\mathbb{Z}/6)^* \cong \mathbb{Z}/4 \times \mathbb{Z}/2$ is $\boxed{\text{not cyclic}}$:

~~(1,1) ≠ (0,0) but has order 2, so it is not a generator~~

no element has order 8, since for any $(a,b) \in \mathbb{Z}/4 \times \mathbb{Z}/2$

$$4 \cdot (a,b) = (0,0).$$

ⓑ $\boxed{\text{Yes}}$ for example, $\varphi : \mathbb{Z}/_{3\cdot 61} \to \mathbb{Z}/_{2013}$

choose generator α ; choose generator β

set $\varphi(\alpha) = \beta^{11}$

(so $\varphi(\alpha^i) = \beta^{11 \cdot i}, \forall i$).

ⓒ the number of 61-Sylow subgroups of a group G of order 2013 divides $\frac{2013}{61} = 33$ and is $\equiv 1 \bmod 61$, so is $= 1$, so the 61-Sylow is normal in G. $\boxed{\text{There doesn't exist a simple group with 2013 elements.}}$

ⓓ $\boxed{\text{Yes}}$, take any rotation around an axis over an angle $\frac{2\pi}{n}$ for some $n \in \mathbb{N}$. It has order n.

Ex. 3. We have to show: if $g \ker\varphi = g' \ker\varphi$ $(*)$ for some $g, g' \in G$, then $\varphi(g) = \varphi(g')$.

From $(*)$, $(g')^{-1} g \in \ker\varphi$. This means, $\varphi((g')^{-1} g) = e_{G'}$

φ is a group hom, so
$$\varphi(g')^{-1}\varphi(g) = e_{G'}$$
so
$$\varphi(g') = \varphi(g).$$

Ex. 4. Write $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, with $r^4 = s^2 = e$ and $rsr = s$.

the following is an isomorphism
$$\varphi : G \longrightarrow D_4$$
(where $G \leq S_4$ is generated by (13) and (1234))
$$(13) \longmapsto s$$
$$(1234) \longmapsto r$$

Indeed, $(13)^2 = (1234)^4 = e$ and
$$(1234)(12)(1234) = (12),$$ so

φ is well-defined & surjective. To prove it is bijective, we check that G contains 8 elements.

e, (1234), $(1234)^2 = (13)(24)$, $(1234)^3 = (1432)$

(13), $(13)(1234) = (12)(34)$, $(13)(1234)^2 = (24)$,

$(13)(1234)^3 = (13)(1432) = (14)(23)$

Ex. 5. (a) $\boxed{U \Rightarrow N \text{ is true:}}$ $\forall g \in G$, gHg^{-1} is a subgroup with m elements, and since it is unique, $gHg^{-1} = H$, so $H \trianglelefteq G$.

(a2) $\boxed{N \Rightarrow U \text{ is false in general}}$. For example, the Klein 4-group $\mathbb{Z}/2 \times \mathbb{Z}/2$ has three normal subgroups $\langle(1,0)\rangle$, $\langle(0,1)\rangle$, $\langle(1,1)\rangle$ of the same order 2 (all are normal, since the 4-group is abelian).

(b) If H' is another normal subgroup with m elements, consider the group homomorphism with kernel H:

$$\varphi : G \twoheadrightarrow G/H$$

Then $\varphi(H')$ has order dividing both $\# H'$ and $\# G/H = [G:H]$, so the coprime hypothesis implies $\varphi(H') = \{e\}$. Hence $H' \leq \ker \varphi = H$. Since $\# H' = \# H$, we find $H' = H$.

Ex.6. (a) A conjugacy class is an orbit for the action of G on itself by conjugation. The number of elements in the orbit of $x \in G$ is

$$[G : G_x] = \frac{\# G}{\# G_x}$$

where $G_x \leq G$ is the stabilizer of x. Hence this order divides $\# G$.

(b) There is always the conjugacy class of $\{e\}$, with one element. Let n denote the order of the group, and m_1, m_2 the orders of the two other conjugacy classes. Then

$$n = 1 + m_1 + m_2$$

and $m_1 | n$, $m_2 | n$.

Hence $m_1 \mid m_1+m_2+1$ so $m_1 \mid m_2+1$ (*)

and similarly, $m_2 \mid m_1+1$ (**)

Assume WLOG that $m_1 \leq m_2$. (***)

We find $m_2 \underset{(**)}{\leq} m_1+1 \underset{(***)}{\leq} m_2 + 1$

so either $m_2 = m_1+1$ or $m_2 = m_1$

From (*), we get $\Downarrow$ $m_1 \mid m_1+2$ or $\Downarrow$ $m_1 \mid m_1+1$

Case $m_1 \mid m_1+2$:

$\Downarrow$

$m_1 \mid 2$

$\Downarrow$

$m_1 = 1$ & $m_2 = 2$ or $m_1 = 2$ & $m_2 = 3$

- $m_1 = 1$ & $m_2 = 2$ $\Rightarrow$ $n = 4$, so $G \cong \mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ $\Rightarrow$ G abelian $\Rightarrow$ all conj. classes in G have one element, contradicting $m_2 = 2$.

- $m_1 = 2$ & $m_2 = 3$ $\Rightarrow$ $n = 6$ & G non-abelian $\Rightarrow$ $\boxed{G \cong D_3}$ $(\cong S_3)$

  and indeed, there are 3 conj. classes, by cycle types

  e, (12), (123)

  $e \to$ 1 elt.; $(12) \to$ 3 elts. $= m_2$; $(123) \to$ 2 elts. $= m_1$

Case $m_1 \mid m_1+1$:

$\Downarrow$

$m_1 = 1 (= m_2)$

$\to$ G has 3 elements, each with trivial conj. class, so $G = Z(G)$, so G is abelian with 3 elements, so $\boxed{G \cong \mathbb{Z}/3}$

<u>Conclusion</u>: $G \cong \mathbb{Z}/3$ or $G \cong D_3$ ∎