Group theory – answers

November 7, 2016

Clearly write your name and student number above each page you hand in. A calculator, phone, books, notes, old exercises et cetera are not allowed. You may use the results (not the exercises) in Armstrong's book to answer the questions unless a result is explicitly asked for. Finally: recall that a group G is called simple if the only normal subgroups of G are $e \in G$ and G itself.
Total points: 90

Statistics: participants 87; avarage grade 6.1; passed: 71%; failed: 29%.

Exercise	1	2	3	4	5	total
Max score	16	42	16	8	8	90
Average score	13.28	22.57	11.71	2.25	1.93	51.75

The following includes take home exercises, is positive.

Possible grades	1	2	3	4	5	6/6.5	7/7.5	8/8.5	9/9.5	10
Grade distribution	3	0	5	6	12	19	33	6	3	0

Exercise 1: Permutation groups and dihedral groups

1. (4pt) Let $\sigma = (123 \ldots 100)$ be an element of S_{100}. Write σ^4 as a product of disjoint cykels. **Answer:** $\sigma^4 = (159 \ldots 97)(2610 \ldots 98)(3711 \ldots 99)(4812 \ldots 100)$.

2. (4pt) Is σ^3 an element of A_{100}? Motivate your answer. **Answer:** No. σ is a 100-cykel and hence it is odd. Therefore σ^3 is a product of 3 odd cycles, which is odd again. A_{100} is group group of even permutations so it does not contain σ^3.

3. (4pt) Let D_{37} be the dihedral group generated by the elements s and r with $s^2 = e, r^{37} = e$ and $srs = r^{-1}$. Determine all elements in the conjugacy class of s. Motivate your answer. **Answer:** These are $sr^k, 0 \leq k < 37$. You can see this as $r^l sr^{-l} = sr^{-2l}, 0 \leq l < 37$ gives all these elements. Also $(sr^l)s(sr^l)^{-1} = sr^{2l}$ gives all these elements.

4. (4pt) Prove that D_{37} is isomorphic to a subgroup of D_k if and only if 37 divides k. **Answer:** The only if part follows from Lagrange's theorem. For the if part, write $D_k =< s, r >$ with $s^2 = e, r^k = e, srs = r^{-1}$. Then $< s, r^{k/37} >$ is a subgroup isomorphic to D_{37}.

Exercise 2: True or false?

For each of the following statements: give a proof or a counterexample.

1. (6pt) Let G be a group. Let $x, y \in G$ be elements of finite order. Then xy has finite order.**Answer 1:** False. In D_∞ we have $x = sr$ and $y = sr^2$ as a counterexample. **Answer 2** (suggested by one student): Consider $\mathbb{Z}_2 * \mathbb{Z}_2$, the group of all finite sequences of 0's and 1's in which no 2 consecutive 0's and 1's occur. Multiplication is concatenation with the rule that all occurrences of 11 and 00 cancel to the empty word. The empty word is the identity. Then 0 has order 2, as $00 = e$. Similarly 1 has order 2. But 01 has infinite order as $010101\ldots01$ is never the identity.

2. (6pt) D_{100} contains a subgroup of index 3. **Answer:** False. If $H < D_{100}$ then the index is given by $|D_{100}|/|H|$. As 3 does not divide $|D_{100}| = 200$ this cannot be true.

3. (6pt) Let G be an abelian group. The conjugacy classes of G contain only 1 element. **Answer:** True. G is abelian iff $\forall g, s \in G : gsg^{-1} = s$ iff $\forall s \in G$ the conjugacy class of s contains 1 element.

4. (6pt) Let $H \lhd G$. Then G is isomorphic to $G/H \times H$. **Answer:** Wrong $H :=< r >$ is normal in D_k as it is of index 2. Moreover $D_k/H \simeq \mathbb{Z}_2$. But D_k is not abelian whereas $G/K \times H \simeq \mathbb{Z}_2 \times H$ is. So The latter two groups cannot be isomorphic. **Remark:** The only sensible interpretation of $G/H \times H$ is $(G/H) \times H$ and not $G/(H \times H)$.

5. (6pt) Let G be an *abelian* group and let $H \lhd G$ be the subgroup of G consisting of all elements of finite order in G. Assume that there exists an element $xH \in G/H$ unequal to eH (i.e. unequal to the identity of G/H). Prove or give a counterexample: Then xH generates an infinite cyclic subgroup of G/H. **Answer:** True. Suppose that xH has finite order, say $(xH)^k = eH$. So that $x^k \in H$. Then x^k has finite order (definition of H) so there is an l with $x^{kl} = e$. But then $x \in H$ (definition of H) so that xH is the identity. Contradiction. We proved that x does not have finite order, hence it generates on infinite cyclic group.

6. (6pt) Let G be a group with $|G| = 42$. Let X be a set with $|X| = 15$. There exists an action of G on X which is transitive. **Answer:** False. The number of elements in an orbit must divide $|G|$. Transitive means that there is only 1 orbit, but 42 does not divide 15.

7. (6pt) There exists a simple group of order $3 \cdot 5 \cdot 59$. **Answer:** False. Take a subgroup H of order 59 (Sylow theorem). The number of such subgroups divides $3 \cdot 5$ and is 1 modulo 59. So there is only 1 subgroup of order 59. Therefore for all $g \in G$ we have $gHg^{-1} = H$. So H is normal in G. So G is not simple.

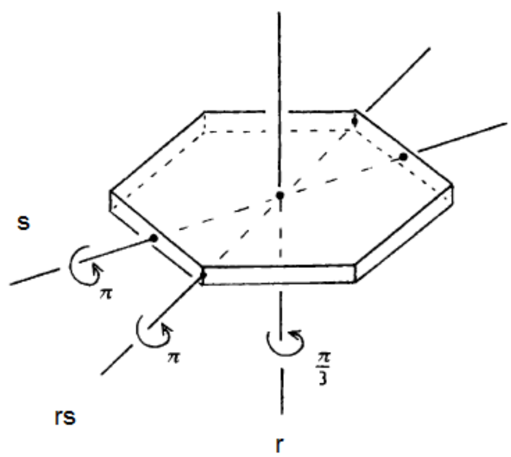
Exercise 3: The counting theorem

(16pt) You want to color the faces of a plate with basis a regular hexagon in two colors (say red and blue). Use the counting theorem to find the number of possible paintings. Two paintings are equal if one can be obtained from the other through turning the plate. You may use the following figure from Armstrongs book to motivate your answer. You may also use that the conjugacy classes of $D_6 =< s, r >$ with $s^2 = e, r^6 = e, srs = r^{-1}$ are given by $\{e\}, \{r, r^5, \}, \{r^2, r^4\}, \{r^3\}, \{s, sr^2, sr^4\}, \{sr, sr^3, sr^5\}$. Explicitly formulate the counting theorem in your answer and show how it is applied. Also motivate how you obtain the numbers in your computation. **Answer:** The number of fixed points for the respective conjugacy classes is: $2^8, 2^3, 2^4, 2^5, 2^4, 2^5$. We compute then

$$\frac{1}{12}\left(2^8 \times 1 + 2^3 \times 2 + 2^4 \times 2 + 2^5 \times 1 + 2^4 \times 3 + 2^5 \times 3\right) = \frac{480}{12} = 40.$$

If you only counted the faces on the side the computation gets:

$$\frac{1}{12}\left(2^6 \times 1 + 2^0 \times 2 + 2^1 \times 2 + 2^3 \times 1 + 2^4 \times 3 + 2^3 \times 3\right) = \frac{156}{12} = 13.$$



Exercise 4: Distinguishing groups

(8pt) The groups D_∞ and $D_\infty \times D_\infty$ are both infinite. Show however that these groups are not isomorphic. **Answer 1:** $[D_\infty, D_\infty]$ is the group generated by r^2. So $D_\infty/[D_\infty, D_\infty]$ has 4 elements. Let $H = D_\infty \times D_\infty$. Then $H/[H, H]$ has 4^2 elements. So the groups cannot be isomorphic because they have non-isomorphic abelianizations. **Answer 2:** $D_\infty \times D_\infty$ contains a subgroup isomorphic to $\mathbb{Z}^2$ whereas D_∞ does not. **Answer 3:** $D_\infty \times D_\infty$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and D_∞ does not. **Answer 4:** $D_\infty \times D_\infty$ 2 elements of order 2 that commute and (after a little check) one sees that D_∞ does not.

Exercise 5: Sylow theorems

(8pt) Show that every group of order $5^2 \times 17 \times 37$ is abelian. **Answer 1:** Let G be a group with order $5^2 \times 17 \times 37$. Let H_{25}, H_{17} and H_{37} be Sylow subgroups corresponding to the primes 5, 17 and 37. The primes are chosen in such a way that the Sylow theorems give us that there are unique such subgroups H_{25}, H_{17} and H_{37}. Let $a \in H_{25}, b \in H_{17}, c \in H_{37}$. Then as $aH_{17}a^{-1} = H_{17}$ we see that $aba^{-1} \in H_{17}$. H_{17} is cyclic so, say that $aba^{-1} = b^k$. Then as $a^{25} = e$ we see that for all l we get $b = a^{25l}ba^{-25l} = b^{k^{25l}}$. This can only happen if k^{25l} is 1 modulo 17. Therefore we may see that k and 17 are relatively prime. But if we choose l such that $25l = 1$ mod 16 we get that $k^{25l} = k$ mod 17 (see Armstrong 11.5). So modulo 17 there is only one choice for k, namely 1. This shows that a and b commute. The same method shows that a and c commute. So our three Sylow groups mutually commute. In Armstrong we proved that there are only 2 groups of order 25, namely $\mathbb{Z}_{25}$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ which are abelian. The groups H_{17} and H_{37} are cyclic hence abelian. So we conclude: H_{25}, H_{17} and H_{37} are abelian groups that mutually commute and generate G. So G is abelian.

Answer 2: (Largely the same) Let G be a group with order $5^2 \times 17 \times 37$. Let H_{25}, H_{17} and H_{37} be Sylow subgroups corresponding to the primes 5, 17 and 37. The primes are chosen in such a way that the Sylow theorems give us that there are unique such subgroups H_{25}, H_{17} and H_{37} and hence they are normal. Let $a \in H_{25}, b \in H_{17}, c \in H_{37}$. Then as $aH_{17}a^{-1} = H_{17}$ and $bH_{25}b^{-1} = H_{25}$ we see that $aba^{-1}b^{-1} \in H_{17} \cap H_{25}$. From Lagrange's theorem we must have $H_{17} \cap H_{25} = \{e\}$. So $aba^{-1}b^{-1} = e$. So a and b commute. The same argument shows that a and c commute and that b and c commute. So our three Sylow groups mutually commute. In Armstrong we proved that there are only 2 groups of order 25, namely $\mathbb{Z}_{25}$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ which are abelian. The groups H_{17} and H_{37} are cyclic hence abelian. So we conclude: H_{25}, H_{17} and H_{37} are abelian groups that mutually commute and generate G. So G is abelian.