Exam group theory - Solutions

January 2, 2017

Een Nederlandse versie vind je hiervoor. Clearly write your name and student number above each page you hand in. A calculator, telephone, books, notes or old exercises are not allowed. To answer your questions you may use the results (not the exercises) in the book 'Groups and Symmetry' by Armstrong, unless a result is explicitly asked for. Further: a group G is called simple if the only normal subgroups of G are given by $\{e\}$ with $e \in G$ the identity element, and G itself. You may use that $A_n, n \geq 5$ is a simple group (this is mostly useful for the bonus exercise).

Start every main exercise on a new sheet.

Total number of points: 90. Bonus points: 4.

Exercise 1: Permutation groups and dihedral groups

1. (4pt) Give all elements of D_{10} of order 2. **Answer:** sr^k, r^5 with $0 \leq k \leq 9$.
2. (4pt) Consider D_7 and let $H < K < D_7$ be subgroups such that $H \neq K$ and $K \neq D_7$. Show that H is the trivial group, meaning that H contains only 1 element. **Answer:** As $K < D_7, K \neq D_7$ implies by Lagrange that $|K|$ is either 7 or 2. In both cases $H < K, H \neq K$ implies by Lagrange that $|K| = 1$.
3. (4pt) Let $\sigma_1 = (1\,2\,3\,4)$ and $\sigma_2 = (5\,6\,7\,8)$ be elements of S_8. Give an element $\tau \in S_8$ such that $\sigma_1 = \tau\sigma_2\tau^{-1}$. **Answer:** For example $\tau = (1\,5)(2\,6)(3\,7)(4\,8)$ works. There are many other possibilities too.
4. (4pt) Let $\sigma = (1\,2\,3\,\dots\,50)$ be an element of S_{50}. Write σ^{49} as a product of disjoint cykels. **Answer:** σ^{49} is the inverse of σ so it is $(50\,49\,48\,\dots\,1)$.

Exercise 2: True or false?

Prove or give a counterexample.

1. (6pt) Let G be an *abelian* group. Let $x, y \in G$ be elements of finite order. Then xy has finite order. **Answer:** True. Suppose $x^n = e, x^m = e$. Then (using that G is abelian) we get that $(xy)^{nm} = x^{nm}y^{nm} = ee = e$. So the order of xy is at most nm. Remark: it need not be equal to nm.
2. (6pt) The group $\mathbb{Z}_6 \times \mathbb{Z}_{15}$ is cyclic. **Answer:** False. This is just Theorem 10.1.
3. (6pt) There exists a normal subgroup H of D_{37} such that D_{37}/H is isomorphic to $\mathbb{Z}_2$. **Answer:** True. Take $H =< r >$ (or use Cauchy to assert the existence of a group H with 37 elements). As $|D_{37}| = 2 \cdot 37$ we see that H is an index 2 subgroup. Hence it is normal (Armstrong) and the quotient group has only 2 elements, hence we have $D_{37}/H \simeq \mathbb{Z}_2$.
4. (6pt) There exists a normal subgroup H of S_7 such that S_7/H is isomorphic to $\mathbb{Z}_{11}$. **Answer:** False. If this were to be true then $|\mathbb{Z}_{11}| = |S_7|/|H|$ but 11 is not a divisor of 7!.

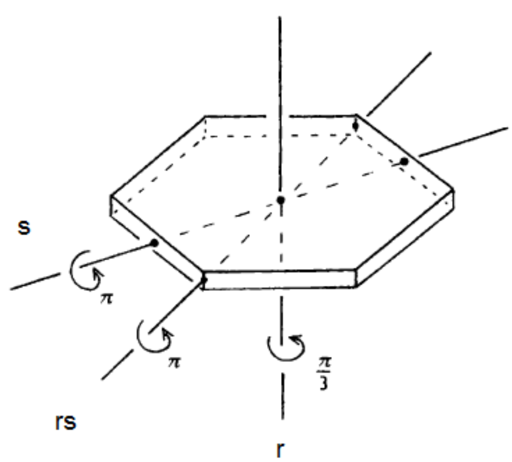
5. (6pt) Let G be a finite group that acts on a set X. For $g \in G$ let $X^g = \{x \in X \mid g(x) = x\}$. Then $|X^g|$ divides $|G|$. **Answer:** False. Exercise 3 gives many counter examples.

6. (6pt) Consider the action of GL_n on $X := GL_n$ by conjugation. Thus $g(x) = gxg^{-1}, g \in GL_n, x \in X$. This action has infinitely many orbits. HINT: Use the determinant in a suitable way. **Answer:** We have $\det(g^{-1}xg) = \det(x)$ so elements in the same orbit have the same determinant. As there are infinitely many possibilities for the determinant, we have infinitely many orbits.

7. (6pt) There exists a simple group of order $7 \cdot 11 \cdot 137$. **Answer:** False. By the first Sylow theorem there exists a subgroup of order 137, say H. The number such subgroups is a divisor of $7 \cdot 11$ which is 1 mod 137. So there is only one such H. As for every $g \in G$ the set gHg^{-1} is actually a subgroup of G of order 137 we see that we must have $gHg^{-1} = H$. So H is normal. Therefore G is not simple.

Exercise 3: The counting theorem

(16pt) Consider the plate with basis a regular hexagon. You want to put an arrow on each of the 6 faces on the side of the plate. The arrow has to be put in the middle and points either upwards or downwards (i.e. in the direction of the top or bottom of the plate). Use the counting theorem to determine the number of possibilities. Two arrow configurations are the same if one can be obtained from the other by rotating the plate. You may use the following figure from Armstrong's book. You may also use that the conjugation classes of $D_6 =< s, r >$ with $s^2 = e, r^6 = e, srs = r^{-1}$ are given by $\{e\}, \{r, r^5, \}, \{r^2, r^4\}, \{r^3\}, \{s, sr^2, sr^4\}, \{sr, sr^3, sr^5\}$. Explicitly formulate the counting theorem in your answer and show how it is applied. Motivate the numbers that occur in your calculation. **Answer:** The counting theorem says that the number of orbits (= number of possible arrow combinations) equals $1/|G| \sum_{g \in G} |X^g|$. We get $\frac{1}{12}(2^6 \cdot 1 + 2 \cdot 2 + 2^2 \cdot 2 + 2^3 \cdot 1 + 0 \cdot 3 + 2^3 \cdot 3) = 9$. Remark: the exercise asks to state the counting theorem. If you only give the sum $1/|G| \sum_{g \in G} |X^g|$ but do not tell what this sum represents (either orbits, or arrow combinations) then 2 points were substracted.



Exercise 4: Distinguishing groups

(8pt) The groups $D_2 \times D_3 \times \ldots \times D_7$ and $S_7 \times (\mathbb{Z}_2)^6$ both have $2^6 \cdot (7!)$ elements. Show that these groups are however not isomorphic. Here we mean by definition $(\mathbb{Z}_2)^6 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. **Answeer:** There are many possibilities, and here are a couple. Let $G = D_2 \times D_3 \times \ldots \times D_7$ and $H = S_7 \times (\mathbb{Z}_2)^6$. **Answer 1:** The number of elements of order 7 in G is 6, the number of elements of order 7 in H is 6!. **Answer 2:** The number of elements of order 5 in G is 4 and the number of elements of order 5 in G is much more (write down a couple, it is not hard; the precise number is $\binom{7}{2}4!$). **Answer 3:** Whenever you take an element $x \in G$ of order 5 and an element $y \in G$ of order 7 we have that x and y commute. This is not true in H as a 5-cykel and a 7-cykel in fact never commute. **Answer 4:** The commutator subgroup of G is isomorphic to $\{e\} \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_7$ which is abelian. The commutator subgroup of H is $A_7 \times \{e\}$ which is not abelian.

Exercise 5: Counting homomorphisms

Let $n \geq 5$ and $k \geq 3$. Assume that k is not divisible by 3.

1. (4pt) Let $\varphi : S_n \to D_k$ be a homomorphism. Prove that A_n is contained in the kernel of φ. Hint: What can be said about the image under φ of a 3-cykel in S_n? **Answer:** D_k does not contain any elements of order 3. As for a 3-cykel $\sigma \in S_n$ we must have $\varphi(\sigma)^3 = \varphi(\sigma^3) = \varphi(e) = e$ we see that the only possibility for the image of σ under φ is the identity of D_k. So all 3-cykels are in the kernel of φ. As the 3-cykels generate A_n (see Armstrong) we get that $A_n < \ker(\varphi)$.

2. (4pt) How many homomorphisms $S_n \to D_k$ are there? Remark: The answer depends on k, but is independent of n. **Answer:** By the previous exercise the image of a homomorphism is completely determined by the image of the 2-cykel $(1\,2)$, because then indeed all elements in A_n must map to the identity and all elements of $(1\,2)A_n$ must have the same image as $(1\,2)$. The image of $(1\,2)$ must either be the identity or an element of order 2, call this element x. There are $k+1$ choices for such x in D_k in case k is odd, and $k+2$ such elements in case k is even (this is more or less exercise 1.1). We need to show that mapping $(1\,2)A_n$ to x and A_n to e is a homomorphism. But this is easy, because this mapping is the composition of the quotient map $S_n \to S_n/A_n \simeq \mathbb{Z}_2$ and the homomorphism $\mathbb{Z}_2 \to D_k$ that sends 1 to x.

3. (Bonus: 4pt) Now let $n \geq 5$ and $k \geq 3$ be arbitrary, so k may be divisible by 3. How many homomorphisms $S_n \to D_k$ are there? **Answer:** We first show that 5.1 still holds, also if k is divisible by 3. Restrict φ to a map $A_n \to D_k$. As A_n is simple and the kernel of (the restriction of) φ is a normal subgroup of A_n we see that either $\ker(\varphi) = e$ or $\ker(\varphi) = A_n$. If $\ker(\varphi) = e$ then φ is injective and so A_n would be a normal subgroup of D_k. However, in D_k any 2 elements of order 3 would commute (as they are in $< r >$) and in A_n this is not true. So we conclude that this is nonsense and we must have that $\ker(\varphi) = A_n$. So we showed that exercsise 5.1 still holds. Then one can copy the answer of 5.2 verbatim to the case of exercise 5.3 to conclude.