Group Theory Retake WISB221 on Thursday 2 Jan 2014, 13:30 - 16:30

Universiteit Utrecht

De Nederlandse versie van het tentamen staat op pagina 1/2. You can do the exam in Dutch or in English. The questions are displayed in Dutch on page 1/2 and in English on page 3/4.

Write your name and student number on every solution page that you hand in. There is a total of 5 exercises; use a new page for every exercise.

Don't just provide the answer, but also show the reasoning that leads to the answer.

You are *not* allowed to use calculators, computers, phones, books, handouts and notes, etc. You are allowed to use the following factorisation: $2014 = 2 \cdot 19 \cdot 53$.

Exercise 1.

6pt (a) What is the sign of the permutation

$$\sigma := (1\,2)(3\,4)\cdots(2013\,2014)$$

in S_{2014}?

6pt (b) Rewrite the permutation $(1\,2\,3)(1\,2\,3)(1\,2)(1\,3)$ from S_3 as a product of disjoint cycles.

6pt (c) Representing the dihedral group D_9 as $\{e, r, \ldots, r^8, s, sr, \ldots, sr^8\}$ for $r, s \in D_9$ with $r^9 = s^2 = e$ and $rsr = s$, which of these 18 elements is $r^2 s^1 r^{2014}$?

6pt (d) Determine the order of the following matrix in $GL(3, \mathbb{R})$:

$$m := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

6pt (e) Is the matrix m from Exercise 1(d) an element of $SO(3, \mathbb{R})$?

10pt **Exercise 2.** Prove Cayley's theorem: every group is isomorphic to a subgroup of a permutation group.

[Please turn over]

Exercise 3. Prove or disprove:

6pt (a) The group $(\mathbf{R}, +)$ of real numbers for addition is cyclic.

6pt (b) If $\varphi \colon G \to H$ is a group homomorphism between finite groups with non-trivial image (i.e., $\varphi(G) \neq \{e_H\}$), then $\#G$ and $\#H$ have a common divisor > 1.

6pt (c) There exists a simple group of order 2014.

6pt (d) If N_1 is a normal subgroup of N_2, and N_2 is a normal subgroup of N_3, then N_1 is always a normal subgroup of N_3.

6pt (e) If H_1 and H_2 are subgroups of a group G, such that H_1 has order 2014 and H_2 has order 7, then $H_1 \cap H_2 = \{e_G\}$.

Exercise 4. Consider the subset H of the direct product group $G := \mathbf{Z} \times \mathbf{Z}$ defined by $H := \{(a, a) \colon a \in \mathbf{Z}\}$.

3pt (a) Prove that H is a normal subgroup of G.

7pt (b) Prove that the quotient group G/H is isomorphic to $\mathbf{Z}$.

Exercise 5. A group G acts *transitively* on a set X if the action has precisely one orbit.

6pt (a) Let G denote a finite group that acts transitively on a set X. Prove that $\#X$ divides $\#G$.

6pt (b) Give an example of an action of a finite group G on a set X such that $\#X$ does not divide $\#G$.

8pt (c) Suppose that G is a group acting on a set X and H a subgroup of G. Prove that H acts transitively on X if and only if the following two properties hold:

(i) G acts transitively on X;

(ii) there exists an element $x \in X$ such that $G = HG_x$, where G_x is the stabiliser of x in G.

Notation: If K and L are subsets of a group, then

$$KL = \{kl \colon k \in K \text{ en } l \in L\}$$

is the set of products of elements of K with elements of L.

[The End]