Inleiding Topologie, Exam A (April 18, 2012)

Exercise 1. Let $\mathcal{B}$ be the familly of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and the subsets

$$[n, a) := \{r \in \mathbb{R} : n \leq r < a\} \quad \text{with} \quad n \in \mathbb{Z}, a \in \mathbb{R}.$$

1. Show that $\mathcal{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. Denote by $\mathcal{T}$ the associated topology. (*1p*)
2. Is $(\mathbb{R}, \mathcal{T})$ second countable? But Hausdorff? But metrizable? Can it be embedded in $\mathbb{R}^{2012}$ (with the Euclidean topology)? (*1p*)
3. compute the closure, the interior and the boundary of $A = [-\frac{1}{2}, \frac{1}{2}]$ in $(\mathbb{R}, \mathcal{T})$. (*1.5p*)

Exercise 2. Prove directly that the abstract torus T_{abs} is homeomorphic to $S^1 \times S^1$. More precisely, define an explicit map

$$\tilde{f} : [0, 1] \times [0, 1] \to \mathbb{R}^4$$

whose image is

$$S^1 \times S^1 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + t^2 = 1\}$$

and which induces a homeomorphism $f : T_{\text{abs}} \to \mathbb{S}^1 \times S^1$. Provide all the arguments. (*1.5p*).

Exercise 3. Let X be the space obtained from the sphere S^2 by gluing the north and the south pole (with the quotient topology). Show that X can be obtained from a square $[0, 1] \times [0, 1]$ by glueing some of the points on the *boundary* (note: you are not allowed to glue a point in the *interior* of the square to any other point). More precisely:

1. Describe the equivalence relation R_0 on S^2 encoding the glueing that defines X. (*0.25p*)
2. Make a picture of X in $\mathbb{R}^3$. (*0.25p*)
3. Describe an equivalence relation R on $[0, 1] \times [0, 1]$ encoding a glueing with the required properties. (*1p*)
4. Show that, indeed, X is homeomorphic to $[0, 1] \times [0, 1]/R$ (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture). (*0.5p*)

Exercise 4. Show that:

1. There exist continuous surjective maps $f : S^1 \to S^1$ which are not injective. (*0.5p*)
2. Any continuous injective map $f : S^1 \to S^1$ is surjective. (*1p*)

Exercise 5. Show that any continuous map

$$f : (\mathbb{R}, \mathcal{T}_{\text{Eucl}}) \to (\mathbb{R}, \mathcal{T}_l)$$

must be constant (recall that $\mathcal{T}_l$ is the lower limit topology- i.e. the one generated by intervals of type $[a, b)$). (*1.5p*)

Note 1: Motivate all your answers. Whenever you use a Theorem or Proposition, please make that clear (e.g. by stating it). Please write clearly (English or Dutch).

Note 2: The final mark is

$$\min\{10, 1 + p\},$$

where p is the number of points you collect from the exercises.