Inleiding Topologie, Exam A (April 17, 2013))

Note 1: The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 13 points in the game!).

Note 2: please MOTIVATE ALL YOUR ANSWERS (e.g., in Exercise 2, do not just give the example, but also explain/prove why it has the required properties).

+ **Exercise 1.** *(1 pt)* Let X and Y be two topological spaces. For $A \subset X$, $B \subset Y$, we consider $A \times B$ as a subset of $X \times Y$. Show that:

$$\text{Int}(A \times B) = \text{Int}(A) \times \text{Int}(B)$$

(the interior of $A \times B$ inside $X \times Y$ (with respect to the product topology)= the product of the interior of A in X with the interior of B in Y).

+ **Exercise 2.** *(1 pt)* Give an example of a connected, bounded, open subset of $\mathbb{R}^2$ which cannot be written as a finite union of balls (here we use the Euclidean metric and topology on $\mathbb{R}^2$).

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Exercise 3. Let $X = (-1, \infty)$.

+ (i) *(1 pt)* Find all the numbers $a, b \in \mathbb{R}$ with the property that

$$n \cdot t = \phi_n(t) = 2^n t + a^n + b$$

defines an action of the group $(\mathbb{Z}, +)$ on X.

+ (ii) *(1 pt)* For the a and b that you found, show that the resulting quotient space $X/\mathbb{Z}$ is homeomorphic to S^1.

+ **Exercise 4.** On $X = \mathbb{Z}$ we consider the family $\mathcal{B}$ of subsets of X consisting of the empty set and the subsets of type

$$N_{a,b} := a + b\mathbb{Z} = \{a + bn : n \in \mathbb{Z}\} \subset \mathbb{Z},$$

with $a, b \in \mathbb{Z}$, $b > 0$.

(i) *(0.25 pts)* Show that, for $a, a', b, b_1, b_2 \in \mathbb{Z}$ with $b, b_1, b_2 > 0$:

$$N_{a,1} = N_{a,-1} = \mathbb{Z},$$

$$N_{a,b_1 b_2} \subset N_{a,b_1} \cap N_{a,b_2}$$

and one has the following equivalences:

$$a' \in N_{a,b} \iff a \in N_{a',b} \iff N_{a,b} = N_{a',b}.$$

(ii) *(0.5 pts)* Show that $\mathcal{B}$ is not a topology on X.

(iii) *(0.5 pts)* Show that $\mathcal{B}$ is a topology basis on X. Let $\mathcal{T}$ be the induced topology.

(v) *(1 pt)* Compute the interior and the closure of $A := \{-1, 1\}$ in $(X, \mathcal{T})$.

(iv) *(0.5 pts)* Show that $(X, \mathcal{T})$ is Hausdorff.

(vi) *(0.25 pts)* Show that, for any $b \in \mathbb{Z}$, $b > 0$, $\mathbb{Z}$ can be written as a union of b nonempty subsets that belong to $\mathcal{B}$, each two of them being disjoint.

(vii) *(0.5 pts)* Show that any subset of type $N_{a,b}$ is both open and closed in $(X, \mathcal{T})$.

(viii) *(1 pt)* Show that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p-\text{prime number}} N_{0,p}$$

and then, using (vii) and (v), deduce that the set of prime numbers is infinite.

Exercise 5. Consider the 3-sphere S^3 viewed as a subspace of $\mathbb{C}^2$:

$$S^3 = \{(u, v) : u, v \in \mathbb{C}, |u|^2 + |v|^2 = 1\}.$$

Inside the sphere we consider

$$A := \{(u, v) \in S^3 : |v| = \frac{\sqrt{2}}{2}\}.$$

(i) *(1 pt)* Show that $S^3 \setminus A$ has two connected components.

(ii) *(0.5 pts)* Show that the two connected components, denoted X_1 and X_2, satisfy:

$$\overline{X}_1 \cap \overline{X}_2 = \partial(X_1) = \partial(X_2) = A.$$

(where the closures and boundaries are inside the space S^3).

(iii) *(1 pt)* Consider the unit circle and the closed unit disk

$$S^1 = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1\}, \quad D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

By a solid torus we mean any space homeomorphic to $S^1 \times D^2$. Show that

$$f : S^1 \times D^2 \to \mathbb{R}^3, \ f((\alpha, \beta), (x, y)) = ((2 - x)\alpha, (2 - x)\beta, y)$$

is an embedding and indicate on a picture what the image of f is (... motivating the name "solid torus").

(iv) *(1 pt)* Show that $\overline{X}_i$ is a solid torus for $i \in \{1, 2\}$.

(v) *(1 pt)* Deduce that the 3-sphere can be obtained from two disjoint copies of $S^1 \times D^2$ (i.e. two solid tori) by gluing any point $(z_1, z_2) \in S^1 \times S^1$ in the boundary of the first copy with the point (z_2, z_1) in the boundary of the second.