Inleiding Topologie, Exam B (June 26, 2013)

Notes:

- English or Dutch (or both)- doesn't matter, but please write clearly (thanks!).
- Please motivate your answers. For instance, in exercise 4, part (ii), do not forget, after you write the function f, to check that f is well defined, continuous and that it has the desired property.
- In this exam, the sub-points of any exercise do not fully depend on each other. For instance, in the last exercise, you may do (v) without doing (iv) (... but using it).

Exercise 1. Prove that there is no continuous injective map $f : S^2 \longrightarrow S^1$. *(1p)* (warning: there are injective maps from S^2 to S^1!).

Exercise 2. Let $X = (0, \infty)$ and consider the open cover of X

$$\mathcal{U} = \{U_1, U_2, U_3, \ldots\} \quad \text{with } U_n = (0, n).$$

(i) Show that $\mathcal{U}$ does not admit a subcover which is locally finite. *(0.5p)*

(ii) Describe a locally finite refinement of $\mathcal{U}$. *(0.5p)*

Exercise 3. Consider

$$X = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\} \setminus \{(0, 0, 1), (0, 0, -1)\},$$
$$Y = \{(x, y, z \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2\sqrt{x^2 + y^2}\} \setminus \{(0, 0, 0)\}.$$
$$\pi : X \longrightarrow \mathbb{R}^3,\ \pi(u, v, w) = (2u\sqrt{u^2 + v^2}, 2v\sqrt{u^2 + v^2}, 2w\sqrt{u^2 + v^2}).$$

(i) Compute the closure $\overline{Y}$ of Y in $\mathbb{R}^3$ and prove that it is compact. *(1p)*

(ii) Prove that the one-point compactification of X is homeomorphic to $\overline{Y}$. *(1p)*

(iii) Draw a picture of Y and explain the map π on the picture. *(1p)*

Exercise 4. Let X be a Hausdorff, locally compact, 2^{nd} countable topological space and assume that U is an open in $X \times \mathbb{R}$ containing $X \times \{0\}$:

$$X \times \{0\} \subset U \subset X \times \mathbb{R}.$$

The aim of this exercise is to prove that there exists a continuous function $f : X \longrightarrow (0, \infty)$ such that U contains

$$U_f := \{(x, t) \in X \times \mathbb{R} : |t| < f(x)\}.$$

Consider

$$r : X \longrightarrow \mathbb{R},\quad r(x) = \sup\{r \in (0, 1] : \{x\} \times (-r, r) \subset U\}.$$

(i) Show that one can find an open cover $\{V_i : i \in I\}$ of X and a family $\{r_i : i \in I\}$ of strictly positive real numbers (for some indexing set I) such that

$$r(y) > r_i \quad \forall\, y \in V_i,\ \forall\, i \in I. \qquad (0.5p)$$

(note: depending on the argument that you find, I that you construct may be countable, but it may also be "very large"- e.g. "as large as X").

(ii) For $\{V_i : i \in I\}$, $\{r_i : i \in I\}$ as above, use a partition of unity argument to build a continuous function $f : X \longrightarrow (0, \infty)$ such that $U_f \subset U$. *(1.5p)*

(iii) Deduce that if X is actually compact, then f may be choosen to be constant. *(0.5p)*

Exercise 5. In this exercise we work over $\mathbb{R}$. Let X be a compact, Hausdorff topological space, $C(X)$ the space of real-valued continuous functions on X and let

$$\mathcal{A} \subset C(X).$$

be a point-separating subalgebra. The aim of this exercise is to show that the spectrum $X_{\mathcal{A}}$ is homeomorphic to X. The homeomorphism will be provided by the map:

$$F : X \to X_{\mathcal{A}},\ F(x) = \chi_x|_{\mathcal{A}}$$

(the restriction of $\chi_x : C(X) \to \mathbb{R}$ to $\mathcal{A}$, where we recall that χ_x sends f to $f(x)$).

(i) Show that F is continuous. *(0.5p)*

(ii) Show that F is injective. *(0.5p)*

The next five steps are to prove that F is surjective. Let $\chi \in X_{\mathcal{A}}$ be a character of $\mathcal{A}$.

(iii) Show that, if $f, g \in \mathcal{A}$ and $f \geq g$, then $\chi(f) \geq \chi(g)$. *(0.5p)*

(Hint: recall that, in the proof of the Stone-Weierstrass theorem we showed that, for $f \in \mathcal{A}$ with $f \geq 0$, one has $\sqrt{f} \in \overline{\mathcal{A}}$).

(iv) Show that for all $f \in \mathcal{A}$ one has $|\chi(f)| \leq ||f||_{\sup}$. *(0.5p)*

(v) Deduce that for any sequence $(f_n)_{n\geq 1}$ of elements in $\mathcal{A}$, convergent to some $f \in C(X)$, the sequence $(\chi(f_n))_{n\geq 1}$ is convergent. *(0.5p)*

(vi) Deduce that there exists an extension of $\chi : \mathcal{A} \to \mathbb{R}$ to a continuous map

$$\tilde{\chi} : C(X) \to \mathbb{R}. \qquad (0.5p)$$

(vii) And then show that $\tilde{\chi}$ is a character. *(0.5p)*

Finally:

(viii) Conclude that F is a homeomorphism. *(0.5p)*