Inleiding Topologie
retake, August 24, 2011

Exercise 1 Show that the equation

$$x^5 + 7x^2 - 30x + 1 = 0$$

has at least two solutions $x_0, x_1 \in (0, 2)$. (*1 p*)

Exercise 2 Consider the space $\mathcal{C}([0,1])$ of all continuous maps $f : [0,1] \longrightarrow \mathbb{R}$, endowed with the sup-metric. Show that

$$A := \{f \in \mathcal{C}([0,1]) : x^2 \leq e^{f(x)} + sin(f(x)) \leq x \;\; \forall\, x \in [0,1]\}$$

is a closed and bounded subset of $\mathcal{C}([0,1])$. (*1 p*)

Exercise 3 Describe a subspace $X \subset \mathbb{R}^2$ which is connected, whose closure (in $\mathbb{R}^2$) is compact, but with the property that X is not locally compact. (*1 p*)

Exercise 4 Let $G = (0, \infty)$ be the group of strictly positive reals, endowed with the usual product. Find an action of G on $\mathbb{R}^4$ with the property that $\mathbb{R}^4/G$ is homeomorphic to S^3. (*1 p*)

Exercise 5 Let $X = \mathbb{R}^2$ endowed with the product topology $\mathcal{T}_l \times \mathcal{T}_l$, where $\mathcal{T}_l$ is the lower limit topology on $\mathbb{R}$.

a. Describe a countable topology basis for the topological space X. (*0.5 p*)

b. Find a sequence $(x_n)_{n \geq 1}$ of points in $\mathbb{R}^2$ which converges to $(0,0)$ with respect to the Euclidean topology, but which has no convergent subsequence in the topological space X. (*0.5 p*)

c. Compute the interior, the closure and the boundary (in X) of

$$A = [0, 1) \times (0, 1]. \quad (\mathit{1p})$$

(please use pictures!).

Exercise 6 Decide (and explain) which of the following statements hold true:

a. $S^1 \times S^1 \times S^1$ can be embedded in $\mathbb{R}^4$. (*0.5 p*)

b. S^1 can be embedded in $(0, \infty)$. (*0.5 p*)

c. the cylinder $S^1 \times [0, 1]$ can be embedded in the Klein bottle. (*0.5 p*)

d. The Moebius band can be embedded into the projective space $\mathbb{P}^2$. (*0.5 p*)

e. the projective space $\mathbb{P}^3$ can be embedded in $\mathbb{R}^6$. (*0.5 p*)

Exercise 7 Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathcal{R}_p$ the set of reminders modulo p. In other words,

$$\mathcal{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) : \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

For $q \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denoted by $[q] \in \mathcal{R}_p$ the induced equivalence class. Show that:

a. The operations (on $\mathcal{R}_p$) $+$, $\bullet$ and multiplications by scalars given by

$$[q_1] + [q_2] := [q_1 + q_2],\ [q_1] \bullet [q_2] := [q_1 \cdot q_2],\ \lambda[q] := [\lambda q]$$

are well-defined and make $\mathcal{R}_p$ into an algebra. *(0.5 p)*

b. For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathcal{R}_p$ has only one point. *(0.5 p)*

c. For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathcal{R}_p$ is homeomorphic to S^n *(1 p)* .

Note: please motivate all your answers (e.g., in Exercise 6, explain/prove in each case your answer. Or, in Exercise 4 prove that $\mathbb{R}^4/G$ is homeomorphic to S^3).