Exam Inleiding Topologie, WISB243 2017-01-30, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Francesco, 2: Panagiotis) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score through division by 4.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** We consider the Euclidean topology $\mathscr{T}_{\text{eucl}}$ on $\mathbb{R}$ and also the topology $\mathscr{T}_l$ generated by the intervals $[a,b)$, for $a,b \in \mathbb{R},\ a<b$.

1 pt (a) Show that the intervals $[a,b)$ form a basis for the topology $\mathscr{T}_l$.

2 pt (b) Let $p,q \in \mathbb{R}\ p>0$. Show that the map $T:\mathbb{R}\to\mathbb{R},\ x\mapsto px+q$ is a homeomorphism of the topological space $(\mathbb{R},\mathscr{T}_l)$ to itself.

2 pt (c) Show that $\mathscr{T}_{\text{eucl}} \subset \mathscr{T}_l$.

1 pt (d) Show that $(\mathbb{R},\mathscr{T}_l)$ is Hausdorff.

1 pt (e) Suppose $S\subset\mathbb{R}$ is compact for the topology induced by $\mathscr{T}_l$. Show that S is also compact for the topology induced by $\mathscr{T}_{eucl}$.

2 pt (f) Suppose $S\subset\mathbb{R}$ is compact for (the topology induced by) $\mathscr{T}_l$. Show that for every $a\in\mathbb{R}$ the set $(-\infty,a)\cap S$ is also compact in $(\mathbb{R},\mathscr{T}_l)$.

2 pt (g) Show that $[0,1]$ is not compact in $(\mathbb{R},\mathscr{T}_l)$.

2 pt (h) Show that $(\mathbb{R},\mathscr{T}_l)$ is not locally compact.

9 pt total **Exercise 2.** Let Γ be the group of strictly positive real numbers, with the ordinary multiplication. We equip $\mathbb{R}$ with the Euclidean topology and consider that action of Γ on $\mathbb{R}$ given by $(\gamma,x)\mapsto\gamma x$. The quotient space $Y:=\mathbb{R}/\Gamma$ is equipped with the quotient topology. Let $\pi:\mathbb{R}\to Y$ be the natural map.

2 pt (a) Show that Y consists of the three points $a:=\pi(-1)$, $b:=\pi(0)$ and $c:=\pi(1)$.

4 pt (b) Determine the quotient topology $\mathscr{T}_Y$ of $Y=\{a,b,c\}$ explicitly, i.e., give a list of all open subsets of Y.

3 pt (c) Is Y Hausdorff, connected or compact? Prove the validity of each answer.

12 pt total **Exercise 3.** Let X be a locally compact Hausdorff topological space which is the disjoint union of two non-empty open subsets $X_1, X_2 \subset X$. Let $X^+ = X \cup \{\infty\}$ be the one point compactification of X.

3 pt (a) Show that the implication '(1) $\Rightarrow$ (2)' holds for the following statements:

(1) At least one of the sets X_1 and X_2 is compact.

(2) X^+ is not connected.

In the rest of the exercise, assume that both X_1 and X_2 are connected.

3 pt (b) If U is non-empty, open and closed in X show that U contains one of the sets X_1 and X_2.

4 pt (c) Show that the implication '(2) $\Rightarrow$ (1)' holds as well.

2 pt (d) Give an example of a non-connected locally compact Hausdorff space X such that X^+ is connected.

10 pt total **Exercise 4.** Let D be the closed unit disk in $\mathbb{R}^2$ and let $X := D \times \{-1, 1\} \subset \mathbb{R}^3$ be equipped with the restriction of the Euclidean topology on $\mathbb{R}^3$.

1 pt (a) Show that X is compact Hausdorff.

Let $\varphi : X \to \mathbb{R}^3$ be the map given by $\varphi(x, \pm 1) = (x, \pm\sqrt{1 - \|x\|^2})$, where $\| \cdot \|$ is the Euclidean norm. It is clear that φ is continuous.

2 pt (b) Show that the image of φ is the unit sphere S^2 in $\mathbb{R}^3$.

Let A be the set of continuous functions $f : X \to \mathbb{R}$ such that $f(x, -1) = f(x, +1)$ for all $x \in \partial D$. Here ∂D denotes the boundary of D in $\mathbb{R}^2$.

1 pt (c) Show that A is a subalgebra of $C(X)$.

6 pt (d) Show that the topological spectrum $\mathbf{X}_A$ of A is homeomorphic to S^2.