Herkansing Inleiding Topologie, WISB243 2019-04-15, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Aldo and Francesco, 2: Maarten) and the total **number of sheets** handed in.
- Use a **separate sheet** for each exercise!
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **mention this**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, **do continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The grade will be obtained from your total score T by rounding off $\min(T/4, 10)$ to a half integer above 6 or an integer below 6.5.
- You are free to write the solutions either in English, or in Dutch.

Succes !

13 pt total **Exercise 1.** For $\mathbb{R}$ we consider the collection $\mathscr{B}$ of all subsets of the form

$$[p,q) := \{x \in \mathbb{R} \mid p \leq x < q\}, \qquad \text{with } p,q \in \mathbb{Q}, p < q.$$

3 pt (a) Show that $\mathscr{B}$ is not a topology, but it is a topology basis. Denote by $\mathscr{T}$ the topology generated by $\mathscr{B}$.

2 pt (b) Show that $\mathscr{T}$ contains the Euclidean topology.

2 pt (c) Is $(\mathbb{R}, \mathscr{T})$ first countable, second countable? Is it Hausdorff?

3 pt (d) Show that $[0,1)$ is open and closed in $(\mathbb{R}, \mathscr{T})$. Is $(\mathbb{R}, \mathscr{T})$ connected?

3 pt (e) Show that $[0,1)$ and $[0,1]$ are not compact in $(\mathbb{R}, \mathscr{T})$.

9 pt total **Exercise 2.** Let $X := [0,1] \times [-2,2]$ and the subset $B := [0,1] \times [-1,1] \subset X$ both be equipped with the topology induced by the Euclidean topology on $\mathbb{R}^2$.

We equip X with the equivalence relation R whose equivalence classes are given by $\{(s,t)\}$ for $0 < s < 1$ and $-2 < t < 2$, $\{(s,-2),(s,2)\}$ for $0 < s < 1$; $\{(0,t),(1,-t)\}$ for $-2 < t < 2$ and, finally, $\{(0,\pm 2),(1,\pm 2)\}$. Accordingly, X/R equipped with the quotient topology, is the Klein bottle (which is known to be a Hausdorff space). The associated quotient map is denoted by $\pi : X \to X/R$.

3 pt (a) The restriction of R to B is the relation R_B defined by $b_1 R_B b_2 \iff \pi(b_1) = \pi(b_2)$, for $b_1, b_2 \in B$. Show that R_B is an equivalence relation on B and explicitly determine the associated equivalence classes in B.

1 pt (b) The quotient B/R_B is equipped with the quotient topology. To which well known space is this quotient homeomorphic? (You need not justify your answer.)

Let $\pi_B : B \to B/R_B$ be the associated quotient map.

2 pt (c) Show that there exists a unique map $f : B/R_B \to X/R$ such that for all $b \in B$ we have $f(\pi_B(b)) = \pi(b)$.

3 pt (d) Prove that the map f is an embedding.

11 pt total **Exercise 3.** Let M be a topological space, and assume that $\gamma \mapsto \varphi_\gamma$ is an action of the group $\mathbb{Z}_2 = \{-1, 1\}$ on M by homeomorphisms. Let $M/\mathbb{Z}_2$ be the associated quotient (equipped with the quotient topology), and $\pi : M \to M/\mathbb{Z}_2$ the quotient map.

2 pt (a) Given a subset $V \subset M$, show that $\pi^{-1}(\pi(V)) = V \cup \varphi_{-1}(V)$.

2 pt (b) Show that for $V \subset M$ open, the set $\pi(V)$ is open in $M/\mathbb{Z}_2$.

3 pt (c) Let $\{U_i\}_{i \in I}$ be an open cover of M. For every $m \in M$ let $i_m, j_m \in I$ be indices such that $m \in U_{i_m}$ and $\varphi(m) \in U_{j_m}$. Show that there exists an open neighborhood V_m of m such that $V_m \subset U_{i_m}$ and $\varphi(V_m) \subset U_{j_m}$.

4 pt (d) Show that M is compact if and only if $\pi(M)$ is compact. Hint: for one of the implications consider the collection $\{\pi(V_m)\}$ resulting from (c).

11 pt total **Exercise 4.** For M a locally compact Hausdorff space we denote by $C_c(M)$ the real linear space of continuous functions $M \to \mathbb{R}$ with compact support. If U is an open subset of M, we put $C_c(U) := \{f \in C_c(M) \mid \mathrm{supp} f \subset U\}$.

2 pt (a) If U is open in M, $f \in C_c(M)$, $\psi \in C(M)$ and $\mathrm{supp}\,\psi \subset U$, show that $\psi f \in C_c(U)$.

By a positive integral on an open subset U of M we mean a linear map $I : C_c(U) \to \mathbb{R}$ such that for all $f \in C_c(U)$ we have:

$$f \geq 0 \Rightarrow I(f) \geq 0.$$

A positive integral I on U is said to be strictly positive if for all $f \in C_c(U)$ we have

$$f \geq 0,\ I(f) = 0 \ \Rightarrow f = 0.$$

1 pt (b) Prove the following result. If I is a positive integral on an open subset U of M and $\psi : M \to \mathbb{R}$ is continuous function with $\psi \geq 0$ and $\mathrm{supp}\psi \subset U$ then $I_\psi : f \mapsto I(\psi f)$ is a positive integral on M.

5 pt (c) Assume that M is second countable, and that for every point $m \in M$ there exists an open neighborhood $U_m \ni m$ and a strictly positive integral $I_m : C_c(U_m) \to \mathbb{R}$ on U_m. Show that there exists a strictly positive integral on M.

In the next item you may use without proof that the map $J : C_c((0,1)^n) \to \mathbb{R}$ defined by the n-fold repeated Riemann integral $J(f) = \int_0^1 \cdots \int_0^1 f(x)\, dx_1 \cdots dx_n$, is a strictly positive integral on the open subset $(0,1)^n$ of $\mathbb{R}^n$.

3 pt (d) If M is a topological manifold of dimension $n \geq 1$, show that there exists a strictly positive integral on M.