Herkansing Inleiding Topologie, WISB243 2019-04-15, 13:30 – 16:30

Solution 1

(a) The union of $[0,1)$ and $[2,3)$ does not belong to $\mathscr{B}$, so $\mathscr{B}$ is not a topology. We will show it is a basis. First of all, if $x \in \mathbb{R}$ then $x \in [n, n+1)$ for an integer $n \in \mathbb{Z}$. Hence, $\mathbb{R} = \cup \mathscr{B}$.

Secondly, let $x \in [p_1, q_1) \cap [p_2, q_2)$, where p_1, q_1, p_2, q_2 are rational numbers such that $p_1 < q_1$ and $p_2 < q_2$. Then $p_2 \leq x < q_1$ and $p_1 \leq x < q_2$, hence $x \in [\max(p_1, p_2), \min(q_1, q_2)) \subset [p_1, q_1) \cap [p_2, q_2)$. It follows that $\mathscr{B}$ is a basis.

(b) Since $\mathbb{Q}$ is countable, so is $Q \times \mathbb{Q}$, and it follows that $\mathscr{B}$ is countable. It follows that $(\mathbb{R}, \mathscr{T})$ is second countable, hence also first countable.

(c) The intervals (a,b) with $a, b \in \mathbb{R}$, $a < b$ form a basis $\mathscr{B}_e$ for the Euclidean topology on $\mathbb{R}$. If $x \in (a,b)$, there exist rational numbers $p, q \in \mathbb{Q}$ such that $a < p < x < q < b$, hence $x \in [p,q) \subset (a,b)$ and we see that $\mathscr{T}$ contains $\mathscr{B}_e$ and hence the Euclidean topology.

(d) Since $[0,1) \in \mathscr{B}$ it follows that $[0,1)$ is open for $\mathscr{T}$. Since $[0,1]$ is closed for the Euclidean topology, it is so for $\mathscr{T}$. On the other hand, $\mathbb{R} \setminus [1,2)$ is closed for $\mathscr{T}$, hence $[0,1) = [0,1] \setminus [1,2)$ is closed for $\mathscr{T}$. From what we proved, $[0,1)$ and $\mathbb{R} \setminus [0,1)$ form a partition of $\mathbb{R}$ into sets from $\mathscr{T}$. Therefore, $(\mathbb{R}, \mathscr{T})$ is not connected.

(e) Define $U_n := [1 - \frac{1}{n}, 1 - \frac{1}{n+1})$ for $n \in \mathbb{Z}$, $n \geq 1$. The sets U_n belong to $\mathscr{T}$ and have $[0,1)$ as their union. Since the union is disjoint, the open cover $\{U_n\}_{n \geq 1}$ of $[0,1)$ is infinite, and has no finite subcover. It follows that $[0,1)$ is not compact. Now $[0,1)$ is closed in $\mathbb{R}$ by (d), hence also closed in $[0,1]$. Therfore, the latter set cannot be compact.

Solution 2

(a) The fibers of $\pi|_B : B \to \pi(B)$ form a partition of B. Clearly, R_B is the equivalence relation determined by that partition. Thus, the equivalence classes of R_B are the following fibers:

$$(\pi|_B)^{-1}(\pi(b)) = B \cap \pi^{-1}(\pi(b)) = B \cap R[b], \qquad (b \in B).$$

These are the sets $\{(s,t)\}$ with $0 < s < 1, |t| \leq 1$, and $\{(0,t), (1,-t)\}$ with $-1 \leq t \leq 1$.

(b) From the description of the equivalence classes we see that X/R_B is the Möbius band.

(c) The equivalence classes for R_B are the fibers of the map $g : B \to \pi(B)$ given by $g = \pi|_B$. It follows that there exists a map $\bar{g} : B/R_B \to \pi(B)$ such that $g \circ \pi_B = g$. Let f be the composition of $\bar{g}$ with the inclusion map $i : \pi(B) \to X/R$ (or $\bar{g}$ viewed as map $B/R_B \to X/R$). Then $f \circ \pi_B = i \circ g = \pi_| B$.

(d) If $\xi_1, \xi_2 \in B/R_B$ we write $\xi_j = \pi_B(b_j)$, for suitable $b_1, b_2 \in B$. From $f(\xi_1) = f(\xi_2)$ it follows that $\pi(b_j) = f(\pi_B(b_j)) = f(\xi_j)$ is independent of j. Therefore, $b_1 R_B b_2$ and we conclude $\xi_1 = \xi_2$. It follows that f is injective.

Since π is continuous $X \to X/R_B$, so is its restriction $\pi|_B$ and we see that $f \circ \pi_B : B \to X/R$ is continuous. It now follows from a proven property of the quotient topology on B/R_B that $f : B/R_B \to X/R$ is continuous.

Finally, B is (closed and bounded in $\mathbb{R}^2$ hence) compact and π_B is continuous, hence B/R_B is compact. Furthermore, X/R is Hausdorff. We just proved that $f : B/R_B \to X/R$ is injective continuous. By a well known result it follows that f is an embedding.

Solution 3

(a) Since $\varphi^2 = \mathrm{id}_M$ it follows that φ is bijective with inverse $\varphi^{-1} = \varphi$. It follows that both φ and its inverse are continuous. Hence, φ is a homeomorphism.

Clearly, xRx. If xRy, then either $y = x$ or $y = \varphi(x)$. In the latter case, $\varphi(y) = x$. Hence $x \in \{y, \varphi(y)\}$ and we see that yRx. Finally, if xRy and yRz, then $y \in \{x, \varphi(x)\}$ and $z \in \{y, \varphi(y)\}$. If $y = x$ then $z \in \{x, \varphi(x)\}.If$ y = $\varphi(x)$ then $z \in \{y, \varphi(y)\} = \{\varphi(x), x\}$. In both cases, xRz. If follows that R is an equivalence relation.

Alternative: note that $\Gamma := \{\mathrm{id}_M, \varphi\}$ with composition is a group of homeomorphisms, and $xRy \iff y \in \Gamma x$, so R is an equivalence relation.

(b) Let $x \in \pi^{-1}(\pi(V))$. Then $\pi(x) \in \pi(V)$ or yRx for an element $y \in V$. Hence $x \in \{y, \varphi(y)\} \subset V \cup \varphi(V)$. This shows that $\pi^{-1}(\pi(V)) \subset V \cup \varphi(V)$.

Conversely, if $x \in V \cup \varphi(V)$, then $\pi(x) \in \pi(V) \cup \pi(\varphi(V)) = \pi(V)$. Hence the identity.

(c) If U is open, then $\pi^{-1}(\pi(U))$ is open by (b), hence $\pi(U)$ is open for the quotient topology.

(d) The set $\varphi^{-1}(U_{j_m})$ is open, since φ is continuous. Furthermore, this set contains $\varphi^{-1}\varphi(m) = m$. It follows that $V_m = U_{i_m} \cap \varphi^{-1}(U_{j_m})$ contains m, is open and satisfies the other properties.

(e) If M is compact, then so is $\pi(M) = M/R$ by continuity of M. Conversely, assume that M/R is compact. We will show that M is compact. Let $\{U_i\}_{i \in I}$ be an open covering of M. Then there exist indices i_m and j_m with the properties of (d), since

$\{U_i\}$ is a covering. Let V_m be as in (d). Then the sets $\pi(V_m)$ are open and cover $\pi(M)$. By compactness of the latter, there exists a finite set of points $m_1, \ldots, m_k$ such that

$$\pi(M) \subset \pi(V_{m_1}) \cup \cdots \cup \pi(V_{m_N}).$$

By taking preimages under π we obtain

$$M \subset \cup_{l=1}^{k} \pi^{-1}\pi(V_{m_l}) = \cup_{l=1}^{k}(V_{m_l} \cup \varphi(V_{m_l})) \subset \cup_{l=1}^{k}(U_{i_{m_l}} \cup U_{j_{m_l}})$$

this shows that $\{U_i\}_{i \in I}$ admits a finite subcover. Hence, M is compact.

Solution 4

(a) Clearly, $\psi f \in C(M)$ and $\text{supp}(\psi f) \subset \text{supp}\psi \cap \text{supp} f \subset U$. Since $\text{supp} f$ is compact and $\text{supp}\psi$ closed, it follows that $\text{supp}\psi \cap \text{supp} f$ is compact, hence $\psi f \in C_c(U)$.

(b) By (a) the map $I_\psi : C_c(M) \to \mathbb{R}$ is well-defined and linear. If $f \geq 0$, then $\psi f \geq 0$, so $I_\psi(f) = I(\psi f) \geq 0$, and we see that I_ψ is a positive integral.

(c) Since M is locally compact and second countable it is paracompact, hence allows partitions of unity. By the assumption, there exists an open cover $\{U_j\}_{j \in J}$ of M and for each $j \in J$ a strictly positive integral on U_j. Let $\{\eta_j\}_{j \in J}$ be a partition of unity subordinate to $\{U_j\}_{j \in J}$. Then by (b), for each $j \in J$ the map $(I_j)_{\eta_j}$ is a positive integral on M. For each $f \in C_c(M)$ we have that only finitely many functions $\eta_j f$ are non-zero and have compact support contained in U_j, so

$$I(f) = \sum_{j \in J} I_j(\eta_j f) = \sum_{j \in J} (I_j)_{\eta_j}(f)$$

is a finite sum of positive real numbers. It readily follows that I is a positive integral on M. If $I(f) = 0$ then each of the terms in the above sum must be zero, hence $\eta_j f = 0$ for all j. It follows that $f = \sum_{j \in J} \eta_j f = 0$. Therefore, I is strictly positive.

(d) Since a topological manifold is locally compact Hausdorff and second countable, all of the above applies. Therefore, we just need to show that for each $m \in M$ there exists an open neighborhood U and a positive integral I on U. There exists an open neighborhood U of m which is homeomorphic to $\mathbb{R}^n$ which in turn is homeomorphic to $V := (0,1)^n$. Let $\chi : U \to V$ be a homeomorphism. The Riemann integral provides a strictly positive integral I_r on V. For $f \in C_c(U)$ we note that $f \circ \chi^{-1} \in C_c(V)$ and we define $I(f) = I_r(f \circ \chi^{-1})$. Then I is readily seen to be linear and positive. If $I(f) = 0$, then $f \circ \chi^{-1} = 0$ hence $f = 0$ on U and since $\text{supp} f \subset U$ it follows that $f = 0$. Thus, I is strictly positive.

Solution 5

(a) The function $\eta_i : X \to \mathbb{R}$ is continuous, and $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$. Therefore, $V_i = \eta_i^{-1}(\mathbb{R} \setminus \{0\})$ is open in X. Let $x \in X$, then $\sum_{i \in I} \eta_i(x) = 1$ (with only finitely many η_i different from zero). It follows that $\eta_i(x) \neq 0$ for at least one i, hence $x \in V_i$. We conclude that $X = \cup_{i \in I} V_i$, hence $\mathscr{V}$ is an open covering of X.

(b) By definition, $\overline{V}_i = \operatorname{supp} \eta_i$. Since $\{\eta_i\}$ is subordinate to $\mathscr{U}$, it follows that $\overline{V}_i = \operatorname{supp} \eta_i \subset U_i$.

(c) Since $V_i \subset \overline{V}_i \subset U_i$, it follows that $\mathscr{V}$ is a refinement. It remains to be shown that $\mathscr{V}$ is locally finite. Let $x \in X$. Since the family $\{\operatorname{supp} \eta_i\}_{i \in I}$ is locally finite, it follows that there exists a neighborhood N of x such that $I_N := \{i \in I \mid \operatorname{supp} \eta_i \cap N \neq \emptyset\}$ is finite. If $V_i \cap N \neq \emptyset$, then $i \in I_N$, so the collection $\mathscr{V}_i$ is locally finite.

(d) First assume (1). Then by a theorem (2) is valid. Now assume (2). Then in the above we have shown that every open covering of X has a locally finite refinement. By definition this implies (1).